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Shapley Value

$$x_i = \sum_{A \subset H(i)} P_n(A)\,[v(A \cup \{i\}) - v(A)]$$

for $H(i) := \{A \subset N : i \notin A\}$

and $P_n(A) := \dfrac{|A|!\,(n-|A|-1)!}{n!}$

What's $H(i)$? Eg. $N = \{1, 2, 3, 4\}$

$H(3)$ = all the subsets of N that $\underline{i}$ is not in

So, $H(3) = \phi, \{1\}, \{2\}, \{4\}, \{1,2\}, \{1,4\}, \{2,4\}, \{1,2,4\}$.

What's $P_n(A)$? E.g. $n = 4$, $A = \{1,3\}$

So, $|A| = 2$, and $P_n(A) = P_4(A) = \dfrac{2!\,(4-2-1)!}{4!}$

$= \dfrac{2!\,1!}{4!} = \dfrac{1}{12}$

Example 16 Winston, p863 § 5.7

Suppose three types of planes (Piper Cubs, DC-10s, and 707s) use an airport. A Piper Cub requires a 100-yd runway, a DC-10 requires a 150-yd runway, and a 707 requires a 400-yd runway. Suppose the cost (in dollars) of maintaining a runway for one year is equal to the length of the runway. Since 707 land at the airport, the airport will have a 400-yd runway. For simplicity, suppose that each year only one plane of each type lands at the airport. How much of the $400 annual maintenance cost should be charged to each plane?

Let Player 1 = Piper Club

" 2 = DC-10

" 3 = 707.

Define: value of a coalition is the cost associated with the runway length needed to service the largest plane in the coalition.

$v(\phi) = \$0$

$v(\{1\}) = \$100$

$v(\{1,2\}) = v(\{2\}) = \$150$

$v(\{3\}) = v(\{1,3\}) = v(\{2,3\}) = v(\{1,2,3\}) = \$400$

Player I's value

S	$P_3(S)$	$v(S \cup \{1\}) - v(S)$
ϕ	2/6	100 − 0
$\{2\}$	1/6	150 − 150
$\{2,3\}$	2/6	400 − 400
$\{3\}$	1/6	400 − 400

$x_1 = \frac{200}{6}$

Player II's value

S	$P_3(S)$	$v(S\cup\{2\}) - v(S)$
ϕ	2/6	150 − 0
$\{1\}$	1/6	150 − 100
$\{3\}$	1/6	400 − 400
$\{1,3\}$	2/6	400 − 400

$$X_2 = \frac{350}{6}$$

Player III's value

S	$P_3(S)$	$v(S\cup\{3\}) - v(S)$
ϕ	2/6	400 − 0
$\{1\}$	1/6	400 − 100
$\{2\}$	1/6	400 − 150
$\{1,2\}$	2/6	400 − 150

$$X_3 = \frac{1850}{6}$$

Shapley Value suggests:
Piper Cub pay \$ 33.33, DC-10 pay \$58.33 and 707 pay \$ 308.33.

Implication of Solution:

All planes that use a portion of the runway should divide equally the cost of the runway.

So, first 100 yd - shared among 3

→ each pays $\frac{\$100}{3} = \$33.33$

Next 150 - 100 yd, shared by DC-10 and 707

→ DC-10 pays $\frac{\$100}{2} + \frac{\$50}{2}$

$= \$58.33$

Next 400 - 150 yd, paid by 707 only.

→ 707 pays $\$250 + \frac{\$100}{3} + \frac{\$50}{2}$

$= \$308.33.$

Thinking exercise

What if there are 10 Piper Cubs, 5 DC-10s, and 2 707s? What should be the Shapley value for each aircraft?