Please note that in the exam you are expected to (at least briefly) explain all steps in your solutions; in what follows, we often give only answers, without detailed solutions and/or explanations. The notation in the present paper is consistent with the one used in 2003 and may slightly differ from the one used in the exam paper 2000.

1. (a) $C_T = (S_T - 20)^+ = \max\{S_T - 20, 0\}$, S_T = stock price at $T = 6$ months.

   (b) $p^* = \frac{1+r-d}{u-d}$ with $u = 22/20 = 0.11$, $d = 18/20 = 0.9$ and $r = 0.02$ (for simplicity's sake; taking into account compounding would yield $r \approx 0.0198$ for a period of 6 months given the annual 4%): $p^* = 0.6$. Assuming that $S_1 = 22$ w.p. p^* and $S_1 = 18$ w.p. $1 - p^*$, we get

   $$\mathbf{E}^* \frac{S_1}{1+r} = \frac{20.4}{1.02} = 20 = S_0,$$

   the martingale property.

   (c) $C = \mathbf{E}^* \frac{(S_1-20)^+}{1+r} = 1.176$.

2. (a) The optimal estimator of X from S is a function $\hat{X} = \hat{X}(S)$ of S such that $\mathbf{E}(X - g(S))^2$ is the smallest for all functions $g(S)$ when $g(S) = \hat{X}(S)$; it is given by $\mathbf{E}(X|\,S)$.

   (b) Find $\arg\min_\alpha \mathbf{E}(X - \alpha S)^2$: as $\mathbf{E}(X - \alpha S)^2 = \mathbf{E}\,X^2 - 2\alpha \mathbf{E}\,XS + \alpha^2 \mathbf{E}\,S^2$, we solve $\frac{d\mathbf{E}(\cdots)^2}{d\alpha} = 0$ to get $\alpha = 1/3$ (using $\mathbf{E}\,X^2 = 2$, $\mathbf{E}\,S^2 = 12$).

   (c) $\hat{X} = \alpha S = S/3 = 4/3$; $\hat{Y} = \mathbf{E}(Y|\,S) = \mathbf{E}(S - X|\,S) = S - \hat{X} = 8/3$.

3. (a) A RV τ is a stopping time with respect to a filtration $\{\mathcal{F}_{t\geq 0}\}$ if, for any $t \geq 0$, the event $\{\tau \leq t\} \in \mathcal{F}_t$. **OST**: If $\{X_t\}_{t\geq 0}$ is a martingale and τ a bounded stopping time for it, then $\mathbf{E}\,X_\tau = \mathbf{E}\,X_0$.

   (b) Let X_n = fortune at time n: $X_n = 20 + \sum_{j=1}^n Y_j$, where $Y_j = \pm 1$ w.p. $1/2$ are i.i.d. RV's. Clearly, $\mathbf{E}\,|X_n| \leq 20 + n < \infty$, and, denoting by $\{\mathcal{F}_n\}$ the "natural filtration" of the process X_n, we get

   $$\mathbf{E}(X_{n+1}|\,\mathcal{F}_n) = \mathbf{E}(X_n + Y_{n+1}|\,\mathcal{F}_n) = X_n + \mathbf{E}(Y_{n+1}|\,\mathcal{F}_n) = X_n + \mathbf{E}\,Y_{n+1} = X_n$$

   using the properties of conditional expectation, the independence of the Y_j's and $\mathbf{E}\,Y_j = -1 \times .5 + 1 \times .5 = 0$: this is a martingale.

   $\tau = \min\{n \geq 0 : X_n = 40 \text{ or } 0\}$: by observing the process up to time t, can say if $\tau = t$ or not, so this is a stopping time.

   OST: $\mathbf{E}\,X_\tau = \mathbf{E}\,X_0 = 20$, and as $X_\tau = 40$ (w.p. p) and $X_\tau = 0$ (w.p. $1-p$), we get $\mathbf{E}\,X_\tau = 40p$, $p = 20/40 = 1/2$.

4. (a) A standard Brownian motion process $\{W_t\}_{t\geq 0}$ is a process in continuous time with continuous trajectories such that $W_0 = 0$ and its increments $W_t - W_s \sim N(0, t-s)$ $(0 \leq s \leq t)$ and are independent for non-overlapping time intervals.

(b) As $\mathbf{E}\, W_t = 0$, for $0 \le s \le t$ we get $\mathrm{cov}(W_s, W_t) = \mathbf{E}\, W_s W_t = \mathbf{E}\, W_s(W_s + (W_t - W_s)) = \cdots = s = \min\{s, t\}$ (don't forget to explain all the steps!!).

(c) First note that $\mathbf{E}\, W_t^2 = t < \infty$, and hence for $X_t = W_t^2 - t$ we have $\mathbf{E}\,|X_t| \le \mathbf{E}\, W_t^2 + t = 2t < \infty$. Next, for $0 \le s \le t$,

$$\mathbf{E}\,(X_t | \mathcal{F}_s) = \mathbf{E}\left[(W_s + (W_t - W_s))^2 - t | \mathcal{F}_s\right] = \cdots = W_s^2 - s^2$$

(don't forget to explain all the steps!!).

5. (a) A process with independent stationary increments. Brownian motion; Poisson process.

(b) A distribution F is said to be infinitely divisible (ID) if, for any $n \ge 1$, there exist i.i.d. RV's $Y_1^{(n)}, Y_2^{(n)}, \ldots, Y_n^{(n)}$ such that $Y_1^{(n)} + Y_2^{(n)} + \cdots + Y_n^{(n)} \sim F$. If $\{X_t\}_{t \ge 0}$ is a Lévy process, then, for any $t > 0$ and $n \ge 1$,

$$X_t = Y_1^{(n)} + Y_2^{(n)} + \cdots + Y_n^{(n)}$$

with $Y_k^{(n)} = X_{tk/n} - X_{t(k-1)/n}$, $k = 1, 2, \ldots, n$, being i.i.d. RV's: so X_t always has an ID distribution.

(c) Use ch.f.'s (or m.g.f.'s) to show that if $X \sim N(\mu, \sigma^2)$, we can take $Y_k^{(n)} \sim N(\mu/n, \sigma^2/n)$.

6. (a) $\mathbf{E}\, X_t = 10e^{0.2^2 t/2} = 10e^{0.02t}$. As $X_t = X_s e^{0.2(W_t - W_s)}$, $\mathbf{E}\,(X_t | \mathcal{F}_s) = X_s e^{0.02(t-s)}$. Meaning: the best (in mean quadratic) predictor (estimator) of the stock price at time t from the prices up to time s.

(b) $dX_t = 0.02 X_t dt + 0.2 X_t dW_t$.

(c) It's a solution to an SDE with coefficients $\mu = 0.02x$, $\sigma = 0.2x$, hence a diffusion.

(d) $P(t, x, y) = \mathbf{P}\,(X_{t+s} \le y | X_s = x) = \mathbf{P}\,(X_t \le y | X_0 = x)$, $p(t, x, y) = \frac{d}{dy} P(t, x, y)$.
BWKE: $p'_t = 0.02xp'_x + 0.02x^2 p''_{xx}$ (note: for $v(s, x) = p(t-s, x, y)$, $v'_s = -p'_t$).
FWKE: $p'_t = -(0.02yp)'_y + 0.02(y^2 p)''_{yy}$.
$X_t \sim LN(\ln x, 0.04t)$ (the formula is given in conditions).

(e) $\mathbf{P}\,(\text{exercise}) = \mathbf{P}\,(\max_{t \le 1} X_t > 15) = \mathbf{P}\,(\max_{t \le 1} W_t > 2.025) = 2\mathbf{P}\,(W_1 > 2.025) \approx 0.05$.

7. (a) $R_t = 0.1t + \int_0^t s\, dW_s$; as the integrand is non-random, the integral has independent Gaussian increments (as the pieces of the trajectory of $\{W_t\}$ on non-overlapping time intervals are independent). Hence R_t also has these properties.

(b) $\mathbf{E}\, R_t = 0.1t$; $V(t) = \mathrm{var}(R_t) = \int_0^t s^2 ds = t^3/3$.

(c) $\mathrm{cov}(R_s, R_t) = \mathbf{E}\,(\int_0^s \times \int_0^t) = \mathbf{E}\,\{\int_0^s \times [\int_0^s + (\int_0^t - \int_0^s)]\}$, then use the independence of increments to get $V(s)$. Computing the correlation: $\mathrm{corr}(R_s, R_t) = \mathrm{cov}(R_s, R_t)/\sqrt{V(s)V(t)}$.

(d) $N(0.1, 1/3)$; $\mathbf{P}\,(R_1 > 0.1) = 0.5$.

8. (a) Use $f(x) = e^{-cx}$ and $(dX_t)^2 = \sigma^2 X_t dt$ to get

$$df(X_t) = -ce^{-cX_t} dX_t + \frac{1}{2} c^2 e^{-cX_t} \sigma^2 X_t dt = -ce^{-cX_t} \sigma \sqrt{X_t}\, dW_t,$$

hence a martingale (as an Itô integral).

(b) Observe that $\{T > t\} = \{X_s > 0 \text{ for all } s \leq t\} \in \mathcal{F}_t$ (and hence also $\{T \leq t\} = \overline{\{T > t\}}$ by the properties of σ-algebras).

(c) Using (a): by the OST, for the bounded stopping time $T \wedge t$ (from (b)), $\mathbf{E}\, e^{-cX_{T \wedge t}} = e^{-cx}$. Observe that

$$e^{-cx} = \mathbf{E}\, e^{-cX_{T \wedge t}} \geq \mathbf{E}\, e^{-cX_{T \wedge t}} \mathbf{1}_{\{T \leq t\}} = \mathbf{E}\, e^{-cX_T} \mathbf{1}_{\{T \leq t\}} = \mathbf{E}\, \mathbf{1}_{\{T \leq t\}} = \mathbf{P}\,(T \leq t)$$

as $X_T = 0$ on $\{T \leq t\}$.