Please note that in the exam you are expected to (at least briefly) explain all steps in your solutions; in what follows, we often give only answers, without detailed solutions and/or explanations. The notation in the present paper is consistent with the one used in 2003 and may slightly differ from the one used in the 2002 exam paper.

1. (a) $C_2 = (S_2 - 20)^+ = \max\{S_2 - 20, 0\}$.

   (b) Use either (i) the valuation formula or (ii) the diagram method.

   (i) $C_0 = \frac{1}{(1+r)^2}\mathbf{E}^* X$, where $X = C_2$ is the payoff.
   Stock price diagram:

$$\begin{array}{ccccc} & & & & 45 \\ & & & \nearrow & \\ & & 30 & & \\ & \nearrow & & \searrow & \\ 20 & & & & 15 \\ & \searrow & & \nearrow & \\ & & 10 & & \\ & & & \searrow & \\ & & & & 5 \end{array}$$

   To compute $\mathbf{E}^*$, need to find $p^* = \frac{1+r-d}{u-d} = \frac{1-0.5}{1.5-0.5} = 0.5$, $1 - p^* = 0.5$. Now $C_0 = p^{*2}X(uu) + 2p^*(1-p^*)X(ud) + (1-p^*)^2X(dd) = 6.25$.

   (ii) Or use the diagram method.

2. (a) Note that $S_0 = 10$. Then can take $f(x) = 10e^x$ and $X_t = -1.5t + 2W_t$ and use Itô's formula.

   (b) As $X_t = 0.1e^{1.5t-2W_t}$, can proceed as in (a).

   (c) Since $Y_t = Z_tX_t$ with $Z_t = K - T^{-1}\int_0^t S_u\,du$, can use the product rule of Itô's calculus (note that Z_t is a differentiable function!) to find dY_t: a routine computation.

3. (a) $m_X(t) = \frac{1}{\sigma\sqrt{2\pi}}\int e^{tx-(x-\mu)^2/2\sigma^2}dx$. When $\mu = 0$: complete the square in the exponential: $tx - x^2/2\sigma^2 = -(x-\sigma^2t)^2/2\sigma^2 + \sigma^2t^2/2$ to get $m_X(t) = e^{\sigma^2t^2/2}$.

   (b) As $X = X_0 + \mu$ (X_0 has zero mean and variance σ^2), $m_{X_0+\mu}(t) = e^{\mu t}m_{X_0}(t) = e^{\mu t+\sigma^2t^2/2}$.

   (c) $m_Z(t) = m_{2(X-Y)}(t) = m_{X-Y}(2t) = m_X(2t)m_{-Y}(2t) = m_X(2t)m_Y(-2t) = e^{(8\sigma^2)t^2/2}$, so $Z \sim N(0, 8\sigma^2)$.

   (d) $\mathbf{E}(Z|\,Y) = 2(\mu - Y)$; $\mathbf{E}(e^{3Z}|\,Y) = e^{-6Y+6\mu+18\sigma^2}$.

   (e) $\mathbf{E}\,e^{iuW} = \exp\{i\mu u - \sigma^2u^2/2 + \lambda(e^{iu} - 1)\}$.

   (f) Yes since $\mathbf{E}\,e^{iuW} = (\mathbf{E}\,e^{iuW_n})^n$, where W_n is a sum of two independent $N(\mu/n, \sigma^2/n)$ and $Po(\lambda/n)$ random variables.

(g) $X_t = \mu t + \sigma W_t + N_t$, where $\{W_t\}$ is a std Brownian motion and $\{N_t\}$ a Poisson process with rate λ independent of $\{W_t\}$.

4. (a) See the lecture notes.

   (b) $\{W_t\}$ is adapted to $\mathbf{F}$ by definition, $\mathbf{E}\,|W_t| < \infty$ as $W_t \sim N(0,t)$ (in fact, can easily compute this expectation!), and for $0 < s < t$,

$$\mathbf{E}\left(W_t \middle| \mathcal{F}_s\right) = \mathbf{E}\left(W_s + (W_t - W_s) \middle| \mathcal{F}_s\right) = \cdots = W_s$$

   (don't forget to explain all the steps!!).

   (c) $\{\tau \leq t\}$ means: the "future" values of S_s ($s \in (t,T]$) don't exceed the maximum on $[0,t]$—and this requires the knowledge of the future, which is not contained in $\mathcal{F}_t$.

   (d) $\mathbf{E}\,W_\tau = \mathbf{E}\,W_0$ provided τ is a bounded stopping time for $\{W_t\}$. As

$$20e^{-\tau+5W_\tau} = S_\tau \geq S_0 = 20e^0 = 20,$$

   must have $-\tau + 5W_\tau \geq 0$, i.e. $W_\tau \geq 0.2\tau$. So

$$0 = \mathbf{E}\,W_0 \neq \mathbf{E}\,W_\tau \geq 0.2\mathbf{E}\,\tau > 0$$

   (as $\tau > 0$ with positive probability). Thus the assertion of the OST doesn't hold and hence τ can't be a stopping time (the only other condition of the theorem is met: $\tau \leq T$ is bounded).

5. (a) $W_t = \sigma^{-1}[\ln(\tilde{S}_t/S_0) - (\mu - r)t]$, i.e. each of $\tilde{S}_t$ and W_t is a function of the other—hence the processes have common "histories".

   (b) That the process is adapted and the expectation is finite is obvious (explain why). For $0 < s < t$,

$$\begin{aligned}\mathbf{E}\left(\tilde{S}_t \middle| \mathcal{F}_s\right) = S_0 e^{(\mu-r)t}\mathbf{E}\left(e^{\sigma W_t} \middle| \mathcal{F}_s\right) &= S_0 e^{(\mu-r)t} e^{\sigma W_s}\mathbf{E}\left(e^{\sigma(W_t - W_s)} \middle| \mathcal{F}_s\right)\\ &= \cdots = \tilde{S}_s e^{(\mu-r+\sigma^2/2)(t-s)},\end{aligned}$$

   hence the conclusion.

   (c) Payoff: $g(S_t) = 10^3\mathbf{1}_{\{S_T > K\}}$.

   Hence by the valuation formula, the time $t = 0$ price of the option is

$$\begin{aligned}&e^{-rT}\mathbf{E}\left(10^3\mathbf{1}_{\{S_T>K\}}\right) = 10^3 e^{-rT}\mathbf{P}\left(S_T > K\right) = \cdots\\ &\quad = 10^3 e^{-rT}\mathbf{P}\left(W_T/\sqrt{T} > [\ln(K/S_0) - (r - \sigma^2/2)T]/\sigma\sqrt{T}\right) = \cdots\end{aligned}$$

6. (a) For $u = u(t,y)$,

$$u'_t = -((4-y)u)'_y + 2u''_{yy} = u + (y-4)u'_y + 2u''_{yy}.$$

   (b) $0 = \pi(y) + (y-4)\pi'(y) + 2\pi''(y)$.

(c) Substituting $\pi(y) = Ce^{-(y-\mu)^2/2\sigma^2}$ in the equation from (b) and calculating

$$\pi' = -\frac{y-\mu}{\sigma^2}\pi, \qquad \pi'' = \frac{(y-\mu)^2}{\sigma^4}\pi - \frac{1}{\sigma^2}\pi,$$

we get

$$0 = \left[1 - \frac{(y-4)(y-\mu)}{\sigma^2} + \frac{2(y-\mu)^2}{\sigma^4} - \frac{2}{\sigma^2}\right]\pi(y).$$

The equation holds when $\mu = 4$, $\sigma^2 = 2$. Therefore the stationary distribution is normal indeed, with the above parameters values.

7. (a) Set $\psi(t,a) = 0$, $\psi(t,b) = 1$, $t \geq 0$; then

$$\begin{aligned} v(s,x) &= \mathbf{E}\left(\psi(\tau, X_\tau) \middle| X_s = x\right) \\ &= \mathbf{P}\left(X_\tau = b \middle| X_s = x\right) = \mathbf{P}\left(X_\tau = b \middle| X_0 = x\right) = V(x) \end{aligned}$$

doesn't depend on s as the process is time-homogeneous. As it solves the BWKE

$$v'_s = -\mu v'_x - \frac{\sigma^2}{2} v''_{xx}$$

we use $v'_s = 0$, $v'_x = V'$, $v''_{xx} = V''$ and $\mu = -1$, $\sigma^2 = 1$ to get

$$0 = V' - \frac{1}{2}V'', \qquad V(a) = 0,\ V(b) = 1.$$

(b) Clearly,

$$X_\tau = \begin{cases} b & \text{with probability } V(x), \\ a & \text{with probability } 1 - V(x). \end{cases}$$

To find $V(x)$, solve the differential equation from part (a). Its characteristic equation is $\frac{1}{2}z^2 - z = 0$, roots: $z_1 = 0$, $z_2 = 2$. Hence the general solution $V(x) = C_1 e^{2x} + C_2$, and using the boundary conditions,

$$V(x) = \frac{e^{2x} - e^{2a}}{e^{2b} - e^{2a}}, \qquad a \leq x \leq b.$$

[Now use this to write down the distribution of X_τ.]

(c) Clearly, $\mathbf{P}\left(\max_{t\geq 0} X_t > b\right) = \lim_{a\to-\infty} V(0)$. Letting $a \to -\infty$, we get $\lim_{a\to-\infty} V(x) = e^{-2(b-x)}$, $x < b$. Hence the answer.

8. (a) BWKE: for $v = v(s,x)$,

$$v'_s = -\mu v'_x - \frac{\sigma^2}{2} v''_{xx} = x(x-1)\left(v'_x + \frac{1}{2}v''_{xx}\right).$$

FWKE: for $u = u(t,y)$,

$$\begin{aligned} u'_t = -(\mu u)'_y + \left(\frac{\sigma^2}{2}u\right)''_{yy} &= -(y(1-y)u)'_y + \frac{1}{2}(y(1-y)u)''_{yy} \\ &= 2(y-1)u + (1 - 3y + y^2)u'_y + \frac{1}{2}y(1-y)u''_{yy}. \end{aligned}$$

(b) Similarly to 7(a), for $V(x) = \mathbf{P}\left(X_\tau = 1 \middle| X_0 = x\right)$ with

$$\tau = \min\{t \geq 0 : \; X_t = 0 \text{ or } X_t = 1\},$$

we get from the BWKE

$$0 = -\mu V' - \frac{1}{2}V'' = -x(1-x)V' - \frac{1}{2}x(1-x)V''.$$

For $x \in (0,1)$ this is equivalent to

$$0 = V' + \frac{1}{2}V'', \qquad V(0) = 0, \; V(1) = 1.$$

Characteristic equation: $\frac{1}{2}z^2 + z = 0$, roots: $z = 0$, $z = -2$, so the general solution: $V(x) = C_1 e^{-2x} + C_2$. From the boundary conditions:

$$V(x) = \frac{1 - e^{-2x}}{1 - e^{-2}}, \qquad x \in (0,1).$$