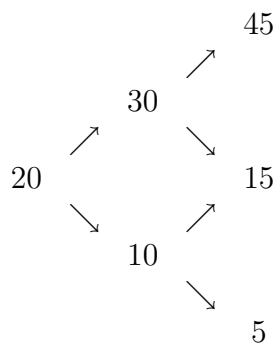


Please note that in the exam you are expected to (at least briefly) explain all steps in your solutions; in what follows, we often give only answers, without detailed solutions and/or explanations. The notation in the present paper is consistent with the one used in 2003 and may slightly differ from the one used in the 2002 exam paper.

1. (a)  $C_2 = (S_2 - 20)^+ = \max\{S_2 - 20, 0\}$ .
- (b) Use either (i) the valuation formula or (ii) the diagram method.

(i)  $C_0 = \frac{1}{(1+r)^2} \mathbf{E}^* X$ , where  $X = C_2$  is the payoff.

Stock price diagram:



To compute  $\mathbf{E}^*$ , need to find  $p^* = \frac{1+r-d}{u-d} = \frac{1+0.5-0.5}{1.5-0.5} = 0.5$ ,  $1 - p^* = 0.5$ . Now  $C_0 = p^{*2}X(uu) + 2p^*(1 - p^*)X(ud) + (1 - p^*)^2X(dd) = 6.25$ .

(ii) Or use the diagram method.

2. (a) Note that  $S_0 = 10$ . Then can take  $f(x) = 10e^x$  and  $X_t = -1.5t + 2W_t$  and use Itô's formula.
  - (b) As  $X_t = 0.1e^{1.5t-2W_t}$ , can proceed as in (a).
  - (c) Since  $Y_t = Z_t X_t$  with  $Z_t = K - T^{-1} \int_0^t S_u du$ , can use the product rule of Itô's calculus (note that  $Z_t$  is a differentiable function!) to find  $dY_t$ : a routine computation.
3. (a)  $m_X(t) = \frac{1}{\sigma\sqrt{2\pi}} \int e^{tx - (x-\mu)^2/2\sigma^2} dx$ . When  $\mu = 0$ : complete the square in the exponential:  $tx - x^2/2\sigma^2 = -(x - \sigma^2 t)^2/2\sigma^2 + \sigma^2 t^2/2$  to get  $m_X(t) = e^{\sigma^2 t^2/2}$ .
  - (b) As  $X = X_0 + \mu$  ( $X_0$  has zero mean and variance  $\sigma^2$ ),  $m_{X_0+\mu}(t) = e^{\mu t} m_{X_0}(t) = e^{\mu t + \sigma^2 t^2/2}$ .
  - (c)  $m_Z(t) = m_{2(X-Y)}(t) = m_{X-Y}(2t) = m_X(2t)m_{-Y}(2t) = m_X(2t)m_Y(-2t) = e^{(8\sigma^2)t^2/2}$ , so  $Z \sim N(0, 8\sigma^2)$ .
  - (d)  $\mathbf{E}(Z|Y) = 2(\mu - Y)$ ;  $\mathbf{E}(e^{3Z}|Y) = e^{-6Y+6\mu+18\sigma^2}$ .
  - (e)  $\mathbf{E} e^{iuW} = \exp\{i\mu u - \sigma^2 u^2/2 + \lambda(e^{iu} - 1)\}$ .
  - (f) Yes since  $\mathbf{E} e^{iuW} = (\mathbf{E} e^{iuW_n})^n$ , where  $W_n$  is a sum of two independent  $N(\mu/n, \sigma^2/n)$  and  $Po(\lambda/n)$  random variables.

- (g)  $X_t = \mu t + \sigma W_t + N_t$ , where  $\{W_t\}$  is a std Brownian motion and  $\{N_t\}$  a Poisson process with rate  $\lambda$  independent of  $\{W_t\}$ .
4. (a) See the lecture notes.
- (b)  $\{W_t\}$  is adapted to  $\mathbf{F}$  by definition,  $\mathbf{E}|W_t| < \infty$  as  $W_t \sim N(0, t)$  (in fact, can easily compute this expectation!), and for  $0 < s < t$ ,

$$\mathbf{E}(W_t | \mathcal{F}_s) = \mathbf{E}(W_s + (W_t - W_s) | \mathcal{F}_s) = \dots = W_s$$

(don't forget to explain all the steps!!).

- (c)  $\{\tau \leq t\}$  means: the “future” values of  $S_s$  ( $s \in (t, T]$ ) don't exceed the maximum on  $[0, t]$ —and this requires the knowledge of the future, which is not contained in  $\mathcal{F}_t$ .
- (d)  $\mathbf{E}W_\tau = \mathbf{E}W_0$  provided  $\tau$  is a bounded stopping time for  $\{W_t\}$ . As

$$20e^{-\tau+5W_\tau} = S_\tau \geq S_0 = 20e^0 = 20,$$

must have  $-\tau + 5W_\tau \geq 0$ , i.e.  $W_\tau \geq 0.2\tau$ . So

$$0 = \mathbf{E}W_0 \neq \mathbf{E}W_\tau \geq 0.2\mathbf{E}\tau > 0$$

(as  $\tau > 0$  with positive probability). Thus the assertion of the OST doesn't hold and hence  $\tau$  can't be a stopping time (the only other condition of the theorem is met:  $\tau \leq T$  is bounded).

5. (a)  $W_t = \sigma^{-1}[\ln(\tilde{S}_t/S_0) - (\mu - r)t]$ , i.e. each of  $\tilde{S}_t$  and  $W_t$  is a function of the other—hence the processes have common “histories”.
- (b) That the process is adapted and the expectation is finite is obvious (explain why). For  $0 < s < t$ ,

$$\begin{aligned} \mathbf{E}(\tilde{S}_t | \mathcal{F}_s) &= S_0 e^{(\mu-r)t} \mathbf{E}(e^{\sigma W_t} | \mathcal{F}_s) = S_0 e^{(\mu-r)t} e^{\sigma W_s} \mathbf{E}(e^{\sigma(W_t - W_s)} | \mathcal{F}_s) \\ &= \dots = \tilde{S}_s e^{(\mu-r+\sigma^2/2)(t-s)}, \end{aligned}$$

hence the conclusion.

- (c) Payoff:  $g(S_T) = 10^3 \mathbf{1}_{\{S_T > K\}}$ .

Hence by the valuation formula, the time  $t = 0$  price of the option is

$$\begin{aligned} e^{-rT} \mathbf{E}(10^3 \mathbf{1}_{\{S_T > K\}}) &= 10^3 e^{-rT} \mathbf{P}(S_T > K) = \dots \\ &= 10^3 e^{-rT} \mathbf{P}\left(W_T / \sqrt{T} > [\ln(K/S_0) - (r - \sigma^2/2)T] / \sigma \sqrt{T}\right) = \dots \end{aligned}$$

6. (a) For  $u = u(t, y)$ ,

$$u'_t = -((4 - y)u)'_y + 2u''_{yy} = u + (y - 4)u'_y + 2u''_{yy}.$$

- (b)  $0 = \pi(y) + (y - 4)\pi'(y) + 2\pi''(y)$ .

(c) Substituting  $\pi(y) = Ce^{-(y-\mu)^2/2\sigma^2}$  in the equation from (b) and calculating

$$\pi' = -\frac{y-\mu}{\sigma^2}\pi, \quad \pi'' = \frac{(y-\mu)^2}{\sigma^4}\pi - \frac{1}{\sigma^2}\pi,$$

we get

$$0 = \left[ 1 - \frac{(y-4)(y-\mu)}{\sigma^2} + \frac{2(y-\mu)^2}{\sigma^4} - \frac{2}{\sigma^2} \right] \pi(y).$$

The equation holds when  $\mu = 4$ ,  $\sigma^2 = 2$ . Therefore the stationary distribution is normal indeed, with the above parameters values.

7. (a) Set  $\psi(t, a) = 0$ ,  $\psi(t, b) = 1$ ,  $t \geq 0$ ; then

$$\begin{aligned} v(s, x) &= \mathbf{E}(\psi(\tau, X_\tau) | X_s = x) \\ &= \mathbf{P}(X_\tau = b | X_s = x) = \mathbf{P}(X_\tau = b | X_0 = x) = V(x) \end{aligned}$$

doesn't depend on  $s$  as the process is time-homogeneous. As it solves the BWKE

$$v'_s = -\mu v'_x - \frac{\sigma^2}{2} v''_{xx}$$

we use  $v'_s = 0$ ,  $v'_x = V'$ ,  $v''_{xx} = V''$  and  $\mu = -1$ ,  $\sigma^2 = 1$  to get

$$0 = V' - \frac{1}{2}V'', \quad V(a) = 0, \quad V(b) = 1.$$

(b) Clearly,

$$X_\tau = \begin{cases} b & \text{with probability } V(x), \\ a & \text{with probability } 1 - V(x). \end{cases}$$

To find  $V(x)$ , solve the differential equation from part (a). Its characteristic equation is  $\frac{1}{2}z^2 - z = 0$ , roots:  $z_1 = 0$ ,  $z_2 = 2$ . Hence the general solution  $V(x) = C_1 e^{2x} + C_2$ , and using the boundary conditions,

$$V(x) = \frac{e^{2x} - e^{2a}}{e^{2b} - e^{2a}}, \quad a \leq x \leq b.$$

[Now use this to write down the distribution of  $X_\tau$ .]

(c) Clearly,  $\mathbf{P}(\max_{t \geq 0} X_t > b) = \lim_{a \rightarrow -\infty} V(0)$ . Letting  $a \rightarrow -\infty$ , we get  $\lim_{a \rightarrow -\infty} V(x) = e^{-2(b-x)}$ ,  $x < b$ . Hence the answer.

8. (a) BWKE: for  $v = v(s, x)$ ,

$$v'_s = -\mu v'_x - \frac{\sigma^2}{2} v''_{xx} = x(x-1) \left( v'_x + \frac{1}{2} v''_{xx} \right).$$

FWKE: for  $u = u(t, y)$ ,

$$\begin{aligned} u'_t &= -(\mu u)'_y + \left( \frac{\sigma^2}{2} u \right)''_{yy} = -(y(1-y)u)'_y + \frac{1}{2} (y(1-y)u)''_{yy} \\ &= 2(y-1)u + (1-3y+y^2)u'_y + \frac{1}{2} y(1-y)u''_{yy}. \end{aligned}$$

(b) Similarly to 7(a), for  $V(x) = \mathbf{P}(X_\tau = 1 | X_0 = x)$  with

$$\tau = \min\{t \geq 0 : X_t = 0 \text{ or } X_t = 1\},$$

we get from the BWKE

$$0 = -\mu V' - \frac{1}{2}V'' = -x(1-x)V' - \frac{1}{2}x(1-x)V''.$$

For  $x \in (0, 1)$  this is equivalent to

$$0 = V' + \frac{1}{2}V'', \quad V(0) = 0, \quad V(1) = 1.$$

Characteristic equation:  $\frac{1}{2}z^2 + z = 0$ , roots:  $z = 0$ ,  $z = -2$ , so the general solution:  $V(x) = C_1 e^{-2x} + C_2$ . From the boundary conditions:

$$V(x) = \frac{1 - e^{-2x}}{1 - e^{-2}}, \quad x \in (0, 1).$$