Solutions to 620-302 Chance and Options Pricing
Exam Paper-2007

1. (a) A European option can only be exercised at maturity, whereas an American option can be exercised at any time prior to or at maturity.

   (b) Assuming, say, continuous time and a constant compounding interest rate r,

   $$P_{A,K}(t) = \max_{\tau} e^{-r\tau}\mathbf{E}^*(S_\tau - K)^- \geq \mathbf{E}^* e^{-rT}(S_T - K)^-$$

   where the maximum is taken over all stopping times (ST's) τ with values in $[0, T]$, and the inequality holds since $\tau \equiv T$ is one such ST.

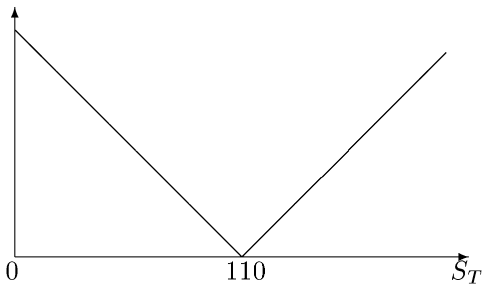
   Bonus question answer: $C_{E,K}(t) \equiv C_{A,K}(t)$, $t \in [0, T]$ (Merton Theorem).

   (c) $P_{E,K_1}(t) < P_{E,K_2}(t)$ since the payoff of the put with strike K_2 is higher: $(S_T - K_1)^- < (S_T - K_2)^-$ for all S_T.

   (d) $C_{E,0}(t) \equiv S_t$ as the payoff of the call is $(S_T - 0)^+ = S_T$, i.e. this is just short-selling of the stock.

2. (a) The market is arbitrage-free as the no-arbitrage condition for binomial markets is satisfied: $d < 1 + r < u$.

   (b) The payoff of the straddle: $g(s) = |s - 110|$.



   (c) Pricing: risk-neutral probabilities are $p^* = \frac{1+r-d}{u-d} = \frac{1.1-0.9}{1.2-0.9} = 2/3$, $q^* = 1 - p^* = 1/3$, and $u^2 S_0 = 144$, $udS_0 = 108$, $d^2 S_0 = 81$. Hence the straddle price is

   $$(1+r)^{-2}\mathbf{E}^* g(S_2) = 1.1^{-2}\big[g(144)(p^*)^2 + g(108)2p^*q^* + g(81)(q^*)^2\big]$$
   $$= \frac{1}{1.21}\big[34(2/3)^2 + 2 \times 2(2/3)(1/3) + 29(1/3)^2\big] \approx 15.886.$$

(d) Stock price diagram:

$$
\begin{array}{ccccc}
 & & & & 144 \\
 & & & \nearrow & \\
 & & 120 & & \\
 & \nearrow & & \searrow & \\
100 & & & & 108 \\
 & \searrow & & \nearrow & \\
 & & 90 & & \\
 & & & \searrow & \\
 & & & & 81
\end{array}
$$

Pricing: using the risk-neutral probabilities $p^* = 2/3$, $1 - p^* = 1/3$ (from part (c)), the straddle price diagram is:

$$
\begin{array}{ccccc}
 & & & & |144 - 110| = 34 \\
 & & & \swarrow & \\
 & & \frac{34\times 2/3+2\times 1/3}{1.1} = \frac{70}{3.3} \approx 21.212 & & \\
 & \swarrow & & \nwarrow & \\
\frac{(70/3.3)\times 2/3+10\times 1/3}{1.1} = \frac{173}{10.89} \approx 15.886 & & & & |108 - 110| = 2 \\
 & \nwarrow & & \swarrow & \\
 & & \frac{2\times 2/3+29\times 1/3}{1.1} \frac{33}{3.3} = 10 & & \\
 & & & \nwarrow & \\
 & & & & |81 - 110| = 29
\end{array}
$$

Answer: the time $t = 0$ price is 15.886; the time $t = 1$ prices are: 21.212 if $S_1 = uS_0$, 10 otherwise.

(e) Put-call parity: $S_0 + P_0 - C_0 = \frac{K}{(1+r)^T}$, where P_0 (C_0) is the time $t = 0$ put (call) price. So

$$C_0 - P_0 = S_0 - \frac{K}{(1+r)^T} = 100 - \frac{110}{1.1^2} = \frac{11}{1.21} \approx 9.091.$$

(f) [i] Self financing trading strategy: $\{(\Delta_t, b_t),\ t = 1, \dots, T\}$, such that $\Delta_t S_t + b_t B_t = \Delta_{t+1} S_t + b_{t+1} B_t$, $t = 1, \dots, T-1$.

[ii] Replication portfolio for a contingent claim: a self financing trading strategy which generates the same cash flow at maturity as the claim.

3. (a) **T**, as $\varphi(t)\overline{\varphi(t)} = |\varphi(t)|^2 \le 1$.

(b) **F**, as $\varphi(t) = \overline{\varphi(t)}$ means that $\varphi(t) \in \mathbf{R}$ (wrong for $X \equiv 1$).

(c) **T**, as $\varphi(2\pi k) = \mathbf{E}\, e^{2\pi kiX} = \mathbf{E}\, 1 = 1$ when X is integer-valued.

(d) **T**, as $\frac{d^2}{dt^2}\mathbf{E}\, e^{itX} = \mathbf{E}\,(-X^2 e^{itX})$.

(e) **F**, as $\varphi(0) + \psi(0) = 2$, impossible for a characteristic function.

(f) **F**, this only means that $\mathbf{E}\, X = 0$.

(g) **F**, as it fails e.g. for $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_1 = \mathcal{F}$: then $\mathbf{E}\,(X|\,\mathcal{F}_0) = \mathbf{E}\, X \neq X = \mathbf{E}\,(X|\,\mathcal{F})$.

(h) **T**, as conditioning on Y and $g(Y)$ is equivalent.

(i) **T**, as $\{\tau = n\} = \bigcup_{k=0}^{n}(\{\tau_1 = k\} \cap \{\tau_2 = n-k\})$, and all the events on the RHS are from $\mathcal{F}_n$.

(j) **T**, as $\{W_t^2 - t\}$ is a martingale, and we can use the optional sampling theorem.

4. (a) For $s, t > 0$, using the properties of conditional expectations: $\mathbf{E}\,(X_{t+s}|\,\mathcal{F}_t) = \mathbf{E}\,(N_{t+s} - N_t + N_t - W_{t+s} + W_t - W_t - 3(t+s)|\,\mathcal{F}_t) =$ [independence of increments] $= \mathbf{E}\,(N_{t+s} - N_t) - \mathbf{E}\,(W_{t+s} - W_t) + \mathbf{E}\,(N_t - W_t|\,\mathcal{F}_t) - 3(t+s) =$ [as N_t, W_t are $\mathcal{F}_t$-measurable] $= \mathbf{E}\, N_s + 0 + N_t - W_t - 3(t+s) = N_t - W_t - 3t = X_t$.

(b) $\mathbf{E}\, e^{itX_2} = \mathbf{E}\, e^{it(N_2 - W_2 - 6)} =$ [by independence] $= e^{-6it}\mathbf{E}\, e^{itN_2} e^{itW_2} = \exp\{-6it + 6(e^{it} - 1) - t^2\}$ [using the formulae at the end of the exam paper: $N_2 \sim$ Poisson(3×2), $W_2 \sim N(0, 2)$].

(c)

$$
\begin{aligned}
\mathbf{E}\,(X_2^2|\,N_2) &= \mathbf{E}\,\big((N_2 - W_2 - 6)^2|\,N_2\big) \\
&= \mathbf{E}\,\big(N_2^2 + W_2^2 + 36 - 2N_2W_2 - 12N_2 + 12W_2|\,N_2\big) \\
&\quad \text{[as } N_t, W_t \text{ are independent + properties of CE's]} \\
&= N_2^2 + 2 + 36 - 2N_2\mathbf{E}\, W_2 - 12N_2 + 12\mathbf{E}\, W_2 \\
&= N_2^2 - 12N_2 + 38. \quad \text{[as } \mathbf{E}\, W_t = 0\text{]}
\end{aligned}
$$

(d) This is again a Gaussian process as vectors $(Y_{t_1}, \ldots, Y_{t_n})$ are linear transformations of $(W_{s_1}, \ldots, W_{s_n})$. So only need to show that the mean & covariance functions coincide with those of the std BM process. Obvious: $\mathbf{E}\, Y_t = t\mathbf{E}\, W_{1/t} = 0$ and, for $0 < s < t$, $\mathbf{E}\, Y_s Y_t = st\mathbf{E}\, W_{1/s}W_{1/t} = st \times (1/t) = s$ as $\min\{1/s, 1/t\} = 1/t$.

(e) $W_1 + 2W_2 - 3W_4 + W_6 = W_1 - 2(W_4 - W_2) + W_6 - W_4 \sim$ [as each group is Gaussian, and due to independent increments] $\sim N(0+0+0, 1 + 2^2 \times 2 + 2) = N(0, 11)$.

(f) Using conditional densities:

$$\begin{aligned} f_{(W_1,W_3)}(x,y) &= f_{W_3|W_1}(y|x) f_{W_1}(x) = f_{W_3 - W_1 + x|W_1)}(y|x) f_{W_1}(x) \\ &= \frac{1}{\sqrt{2\pi}\sqrt{2}} \exp\Big\{-\frac{(y-x)^2}{2\times 2}\Big\} \times \frac{1}{\sqrt{2\pi}} \exp\Big\{-\frac{x^2}{2}\Big\} \\ &= \frac{1}{2\pi\sqrt{2}} \exp\Big\{-\frac{1}{4}y^2 + \frac{1}{2}xy - \frac{3}{4}x^2\Big\}. \end{aligned}$$

5. (a) $S_t = S_0 \exp\{(r - \sigma^2/2)t + \sigma W_t\}$

(b) For $0 \le s < t$, $\mathcal{F}_t = \sigma(W_s,\ s \le t)$,

$$\mathbf{E}\left(e^{-rt} S_t | \mathcal{F}_s\right) = S_0 e^{-\sigma^2 t/2} \mathbf{E}\left(e^{\sigma(W_t - W_s) + \sigma W_s} | \mathcal{F}_t\right) = \text{[by independent increments]}$$
$$= S_0 e^{-\sigma^2 t/2} e^{\sigma W_s} \mathbf{E}\, e^{\sigma(W_t - W_s)} = S_s e^{-rs - \sigma^2(t-s)/2} e^{\sigma^2(t-s)/2} = e^{-rs} S_s$$

(c) We have

$$X_t = 8 S_0^5 \exp\{5(r - \sigma^2/2)t + 5\sigma W_t\}$$
$$= \frac{1}{4}\exp\{5(0.1 - 0.2^2/2)t + 5\times 0.2 W_t\} = \frac{1}{4}\exp\{0.4t + W_t\} = f(Y_t),$$

where $Y_t = 0.4t + W_t$, $f(x) = \frac{1}{4}e^x$, and so $f' = f'' = f$,

$$\begin{aligned} dX_t = f'(Y_t) dY_t + \frac{1}{2} f''(Y_t)(dY_t)^2 &= f(Y_t)\left(0.4dt + dW_t + \frac{1}{2}dt\right) \\ &= 0.225 e^{0.4t + W_t} dt + 0.25 e^{0.4t + W_t} dW_t. \end{aligned}$$

(d)

$$\begin{aligned} C_0 &= e^{-rT} \mathbf{E}^*(S_T - K)^+ = \mathbf{E}\left(e^{-0.02 + 0.2W_1} - e^{-0.02}\right)^+ \\ &= e^{-0.02} \mathbf{E}\left(e^{0.2W_1} - 1; W_1 > 0\right) \\ &= e^{-0.02}\left[\mathbf{E}\left(e^{0.2W_1}; W_1 > 0\right) - \mathbf{P}\left(W_1 > 0\right)\right] \\ &= \frac{e^{-0.02}}{\sqrt{2\pi}} \int_0^\infty e^{0.2x} e^{-x^2/2} dx - \frac{e^{-0.02}}{2} \\ &= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-(x-0.2)^2/2} dx - \frac{e^{-0.02}}{2} \\ &= N(0.2) - \frac{e^{-0.02}}{2} \approx 0.5796 - \frac{e^{-0.02}}{2} \approx 0.0895. \end{aligned}$$

6. (a) By the product rule of Itô's calculus,

$$dX_t = -e^{-t}(x_0-2)dt + \left(\int_0^t e^s dW_s\right)e^{-t}dt - e^{-t}e^t dW_t$$
$$= \left(-e^{-t}(x_0-2) + e^{-t}\int_0^t e^s dW_s\right)dt - dW_t = (2-X_t)dt - dW_t.$$

(b) Since the integrand in the Itô integral non-random, the integral is a normally distributed RV with zero mean and variance $\int_0^t (e^s)^2 ds = (e^{2t}-1)/2$. As X_t is a linear transformation of the integral, we obtain $X_t \sim N\big(2+e^{-t}(x_0-2), (1-e^{-2t})/2\big)$.
As $t \to \infty$, the density converges to that of $N\big(2, 1/2\big)$.

(c) Using Itô's formula with $f(t,x) = e^t x^2$ [so that $f_t = e^t x^2$, $f'_x = 2e^t x$, $f'_{xx} = 2e^t$],

$$dY_t = e^t X_t^2 dt + 2e^t X_t dX_t + \frac{1}{2}2e^t (dX_t)^2$$
$$= [-e^t X_t^2 + 4e^t X_t + e^t]dt - 2e^t X_t dW_t$$
$$= [-Y_t + 4e^{t/2}\sqrt{Y_t} + e^t]dt - 2e^{t/2}\sqrt{Y_t}dW_t.$$

(d) FWKE: for $u = u(t,y)$,

$$u'_t = [-(\mu u)'_y + (\sigma^2 u/2)''_{yy}] = u + (y-2)u'_y + \frac{1}{2}u''_{yy};$$

BWKE: for $v = v(s,x)$,

$$v'_s = [-\mu v'_x - \sigma^2 v''_{xx}/2] = (x-2)v'_x - \frac{1}{2}v''_{xx}.$$

(e) For stationary density $\pi = \pi(y)$,

$$0 = (y-2)\pi + \frac{1}{2}\pi',$$

Solving the equation: $\pi'/\pi = -2(y-2) \Leftrightarrow (\ln\pi)' = -2(y-2)$ $\Leftrightarrow \ln\pi = -(y-2)^2 + C \Leftrightarrow \pi = Ce^{-(y-2)^2}$, which is the density of $N(2, 1/2)$.
This is the same as the limiting distribution found in part (b).

(f) For an appropriate function $\psi(s,x)$, the function $v(s,x) = \mathbf{E}\left[\psi(\tau, X_\tau) | X_s = x\right]$ will satisfy the BWKE. Take $\psi(s,x) = s$. Then

$$v(s,x) = \mathbf{E}\left(\tau | X_s = x\right) = s + \mathbf{E}\left(\tau | X_0 = x\right) = s + U(x).$$

So $v'_s = 1$, $v'_x = U'$, $v''_{xx} = U''$, and hence from the BWKE we get

$$1 = (x-2)U'(x) - \frac{1}{2}U''(x).$$