Solutions to 620-302 Chance and Options Pricing
Exam Paper-2007

1. (a) A European option can only be exercised at maturity, whereas an American option can be exercised at any time prior to or at maturity.

(b) Assuming, say, continuous time and a constant compounding interest rate $r$,

$$P_{A,K}(t) = \max_{\tau} e^{-r\tau} \mathbb{E}^* (S_{\tau} - K)^- \geq \mathbb{E}^* e^{-rT} (S_T - K)^-$$

where the maximum is taken over all stopping times (ST’s) $\tau$ with values in $[0, T]$, and the inequality holds since $\tau \equiv T$ is one such ST.

*Bonus question answer:* $C_{E,K}(t) \equiv C_{A,K}(t)$, $t \in [0, T]$ (Merton Theorem).

(c) $P_{E,K_1}(t) < P_{E,K_2}(t)$ since the payoff of the put with strike $K_2$ is higher: $(S_T - K_1)^- < (S_T - K_2)^-$ for all $S_T$.

(d) $C_{E,0}(t) \equiv S_t$ as the payoff of the call is $(S_T - 0)^+ = S_T$, i.e. this is just short-selling of the stock.

2. (a) The market is arbitrage-free as the no-arbitrage condition for binomial markets is satisfied: $d < 1 + r < u$.

(b) The payoff of the straddle: $g(s) = |s - 110|$.

(c) Pricing: risk-neutral probabilities are $p^* = \frac{1+r-d}{u-d} = \frac{1.1-0.9}{1.2-0.9} = 2/3$, $q^* = 1-p^* = 1/3$, and $u^2S_0 = 144$, $udS_0 = 108$, $d^2S_0 = 81$. Hence the straddle price is

$$(1+r)^{-2} \mathbb{E}^* g(S_2) = 1.1^{-2} \left[ g(144)(p^*)^2 + g(108)2p^*q^* + g(81)(q^*)^2 \right]$$

$$= \frac{1}{1.21} \left[ 34(2/3)^2 + 2 \times 2(2/3)(1/3) + 29(1/3)^2 \right] \approx 15.886.$$
(d) Stock price diagram:

\[ \begin{array}{cccc}
144 & \rightarrow & 120 & \rightarrow \\
100 & \rightarrow & 108 & \rightarrow \\
90 & \rightarrow & 81 & \rightarrow
\end{array} \]

Pricing: using the risk-neutral probabilities \( p^* = 2/3 \), \( 1 - p^* = 1/3 \) (from part (c)), the straddle price diagram is:

\[ |144 - 110| = 34 \]
\[ \frac{34 \times 2/3 + 2 \times 1/3}{1.1} = \frac{70}{3.3} \approx 21.212 \]
\[ \frac{(70/3.3) \times 2/3 + 10 \times 1/3}{1.1} = \frac{173}{10.89} \approx 15.886 \]
\[ |108 - 110| = 2 \]
\[ \frac{2 \times 2/3 + 29 \times 1/3}{3 \times 1.1} = 10 \]
\[ |81 - 110| = 29 \]

Answer: the time \( t = 0 \) price is 15.886; the time \( t = 1 \) prices are: 21.212 if \( S_1 = uS_0 \), 10 otherwise.

(e) Put-call parity: \( S_0 + P_0 - C_0 = \frac{K}{(1+r)^T} \), where \( P_0 \) (\( C_0 \)) is the time \( t = 0 \) put (call) price. So

\[ C_0 - P_0 = S_0 - \frac{K}{(1 + r)^T} = 100 - \frac{110}{1.1^2} = \frac{11}{1.21} \approx 9.091. \]

(f) [i] Self financing trading strategy: \( \{(\Delta_t, b_t), t = 1, \ldots, T\} \), such that \( \Delta_t S_t + b_t B_t = \Delta_{t+1} S_t + b_{t+1} B_t, t = 1, \ldots, T-1. \)

[ii] Replication portfolio for a contingent claim: a self financing trading strategy which generates the same cash flow at maturity as the claim.

3. (a) \( \mathbf{T} \), as \( \varphi(t) \varphi(t) = |\varphi(t)|^2 \leq 1. \)

(b) \( \mathbf{F} \), as \( \varphi(t) = \overline{\varphi(t)} \) means that \( \varphi(t) \in \mathbb{R} \) (wrong for \( X \equiv 1 \)).
4. (a) \( \varphi(2\pi k) = E e^{2\pi kiX} = E 1 = 1 \) when \( X \) is integer-valued.

(d) \( T \), as \( \frac{d}{dt} E e^{itX} = E (-X^2 e^{itX}) \).

(e) \( F \), as \( \varphi(0) + \psi(0) = 2 \), impossible for a characteristic function.

(f) \( F \), this only means that \( EX = 0 \).

(g) \( F \), as it fails e.g. for \( F_0 = \{\emptyset, \Omega\} \), \( F_1 = F \); then \( E(X|F_0) = E X \neq X = E(X|F) \).

(h) \( T \), as conditioning on \( Y \) and \( g(Y) \) is equivalent.

(i) \( T \), as \( \{\tau = n\} = \bigcup_{k=0}^n \{\tau_1 = k\} \cap \{\tau_2 = n - k\} \), and all the events on the RHS are from \( F_n \).

(j) \( T \), as \( \{W_t^2 - t\} \) is a martingale, and we can use the optional sampling theorem.

4. (a) For \( s, t > 0 \), using the properties of conditional expectations:

\[
E(X_{t+s}|F_t) = E(N_{t+s} - N_t + N_t - W_{t+s} + W_t - W_t - 3(t + s)|F_t) = [\text{independence of increments}] = E(N_{t+s} - N_t) - E(W_{t+s} - W_t) + E(N_t - W_t|F_t) - 3(t + s) = [\text{as } N_t, W_t \text{ are } F_t\text{-measurable}]
\]

\[
E N_s + 0 + N_t - W_t - 3(t + s) = N_t - W_t - 3t = X_t.
\]

(b) \( E e^{itX_2} = E e^{it(N_2 - W_2 - 6)} = [\text{by independence}] = e^{-6it} E e^{itN_2} e^{itW_2} = \exp\{-6it + 6(e^{it} - 1) - t^2\} \) [using the formula at the end of the exam paper: \( N_2 \sim \text{Poisson}(3 \times 2), W_2 \sim N(0, 2) \)].

(c)

\[
E(X_2^2|N_2) = E((N_2 - W_2 - 6)^2|N_2)
\]

\[
= E(N_2^2 + W_2^2 + 36 - 2N_2W_2 - 12N_2 + 12W_2|N_2)
\]

[as \( N_t, W_t \) are independent + properties of CE's]

\[
= N_2^2 + 2 + 36 - 2N_2 E W_2 - 12N_2 + 12E W_2
\]

\[
= N_2^2 - 12N_2 + 38. \quad [\text{as } E W_t = 0]
\]

(d) This is again a Gaussian process as vectors \((Y_1, \ldots, Y_n)\) are linear transformations of \((W_{s_1}, \ldots, W_{s_n})\). So only need to show that the mean & covariance functions coincide with those of the std BM process. Obvious: \( E Y_t = tE W_{1/t} = 0 \) and, for \( 0 < s < t \), \( E Y_t Y_s = stE W_{1/s} W_{1/t} = st \times (1/t) = s \) as \( \min\{1/s, 1/t\} = 1/t \).
(e) \( W_1 + 2W_2 - 3W_4 + W_6 = W_1 - 2(W_4 - W_2) + W_6 - W_4 \sim \) [as each group is Gaussian, and due to independent increments] \( \sim N(0 + 0 + 0, 1 + 2^2 \times 2 + 2) = N(0, 11). \)

(f) Using conditional densities:
\[
f_{(w_1,w_2)}(x,y) = f_{w_3 | w_1}(y | x)f_{w_1}(x) = f_{w_2 - w_1 + x | w_1}(y | x)f_{w_1}(x)
\]
\[
= \frac{1}{\sqrt{2\pi}\sqrt{2}} \exp \left\{ -\frac{(y-x)^2}{2 \times 2} \right\} \times \frac{1}{\sqrt{2\pi}} \exp \left\{ -\frac{x^2}{2} \right\}
\]
\[
= \frac{1}{2\pi\sqrt{2}} \exp \left\{ -\frac{1}{4} y^2 + \frac{1}{2} xy - \frac{3}{4} x^2 \right\}.
\]

5. (a) \( S_t = S_0 \exp \{ (r - \sigma^2/2)t + \sigma W_t \} \)

(b) For \( 0 \leq s < t \), \( \mathcal{F}_t = \sigma(W_s, s \leq t) \),
\[
\mathbb{E} \left( e^{-rt} S_t | \mathcal{F}_s \right) = S_0 e^{-\sigma^2 t/2} \mathbb{E} \left( e^{\sigma(W_t - W_s) + \sigma W_t} | \mathcal{F}_t \right) = [\text{by independent increments}]
\]
\[
= S_0 e^{-\sigma^2 t/2} e^{\sigma W_t} \mathbb{E} e^{\sigma(W_t - W_s)} = S_s e^{-r s - \sigma^2(t-s)/2} e^{\sigma^2(t-s)/2} = e^{-rs} S_s
\]

(c) We have
\[
X_t = 8 S_0^5 \exp \{ 5(r - \sigma^2/2)t + 5\sigma W_t \}
\]
\[
= \frac{1}{4} \exp \{ 5(0.1 - 0.2^2/2)t + 5 \times 0.2 W_t \} = \frac{1}{4} \exp \{ 0.4t + W_t \} = f(Y_t),
\]
where \( Y_t = 0.4t + W_t \), \( f(x) = \frac{1}{4} e^x \), and so \( f' = f'' = f \),
\[
dX_t = f'(Y_t) dY_t + \frac{1}{2} f''(Y_t) (dY_t)^2 = f(Y_t) \left( 0.4 dt + dW_t + \frac{1}{2} dt \right)
\]
\[
= 0.225 e^{0.4t + W_t} dt + 0.25 e^{0.4t + W_t} dW_t.
\]

(d)
\[
C_0 = e^{-rT} \mathbb{E}^s (S_T - K)^+ = \mathbb{E} \left( e^{-0.02 + 0.2 W_1} - e^{-0.02} \right)^+
\]
\[
= e^{-0.02} \mathbb{E} \left( e^{0.2 W_1} - 1; W_1 > 0 \right)
\]
\[
= e^{-0.02} \left[ \mathbb{E} (e^{0.2 W_1}; W_1 > 0) - \mathbb{P} (W_1 > 0) \right]
\]
\[
= \frac{e^{-0.02}}{\sqrt{2\pi}} \int_0^\infty e^{0.2x} e^{-x^2/2} dx - \frac{e^{-0.02}}{2}
\]
\[
= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-(x-0.2)^2/2} dx - \frac{e^{-0.02}}{2}
\]
\[
= N(0.2) - \frac{e^{-0.02}}{2} \approx 0.5796 - \frac{e^{-0.02}}{2} \approx 0.0895.
\]
6. (a) By the product rule of Itô’s calculus,
\[ dX_t = -e^{-t}(x_0 - 2)dt + \left( \int_0^t e^s dW_s \right) dt - e^{-t}e^t dW_t \]
\[ = \left( -e^{-t}(x_0 - 2) + e^{-t} \int_0^t e^s dW_s \right) dt - dW_t = (2 - X_t)dt - dW_t. \]

(b) Since the integrand in the Itô integral non-random, the integral is a normally distributed RV with zero mean and variance \( \int_0^t (e^s)^2 ds = (e^{2t} - 1)/2 \). As \( X_t \) is a linear transformation of the integral, we obtain \( X_t \sim N(2 + e^{-t}(x_0 - 2), (1 - e^{-2t})/2) \). As \( t \to \infty \), the density converges to that of \( N(2, 1/2) \).

(c) Using Itô’s formula with \( f(t, x) = e^t x^2 \) [so that \( f_t = e^t x^2, f_x = 2e^t x, f_{xx} = 2e^t \)],
\[ dY_t = e^t X_t^2 dt + 2e^t X_t dX_t + \frac{1}{2} 2e^t (dX_t)^2 \]
\[ = [-e^t X_t^2 + 4e^t X_t + e^t] dt - 2e^t X_t dW_t \]
\[ = [-Y_t + 4e^{t/2} \sqrt{Y_t} + e^t] dt - 2e^{t/2} \sqrt{Y_t} dW_t. \]

(d) FWKE: for \( u = u(t, y) \),
\[ u_t' = \left[ -(\mu u)' + (\sigma^2 u/2)'' \right] = u + (y - 2)u_y' + \frac{1}{2} u_{yy}''; \]

BWKE: for \( v = v(s, x) \),
\[ v_s' = \left[ -\mu v_x' - \sigma^2 v_{xx}' / 2 \right] = (x - 2)v_x' - \frac{1}{2} v_{xx}' \]

(e) For stationary density \( \pi = \pi(y) \),
\[ 0 = (y - 2)\pi + \frac{1}{2} \pi'; \]
Solving the equation: \( \pi'/\pi = -2(y - 2) \Leftrightarrow (\ln \pi)' = -2(y - 2) \Leftrightarrow \ln \pi = -(y - 2)^2 + C \Leftrightarrow \pi = Ce^{-(y-2)^2} \), which is the density of \( N(2, 1/2) \).
This is the same as the limiting distribution found in part (b).

(f) For an appropriate function \( \psi(s, x) \), the function \( v(s, x) = E [\psi(\tau, X_\tau) | X_s = x] \) will satisfy the BWKE. Take \( \psi(s, x) = s \). Then
\[ v(s, x) = E (\tau | X_s = x) = s + E (\tau | X_0 = x) = s + U(x). \]
So \( v'_s = 1, \ v'_x = U', \ v''_{xx} = U'' \), and hence from the BWKE we get

\[
1 = (x - 2)U'(x) - \frac{1}{2}U''(x).
\]