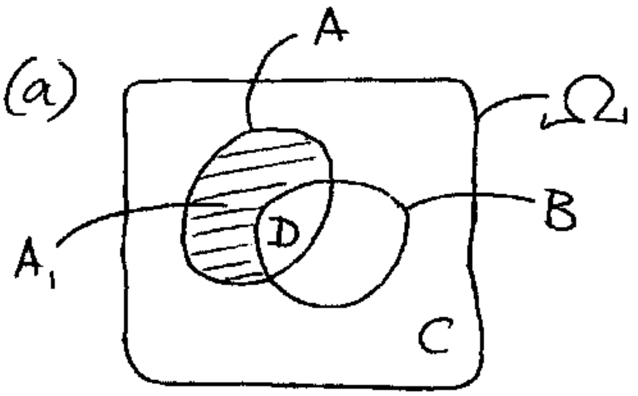
Solutions to HW-3

1 (a)



$\mathcal{F}$ consists of all possible unions of the following events:

$$A_1 = A \cap \bar{B},\quad A_2 = B \cap \bar{A},\quad A_3 = D = A \cap B,\quad A_4 = C = \overline{A \cup B}$$

See the diagram: they all are obtained from A, B using the operations $\bar{\ }, \cap, \cup$ – so all must $\in \mathcal{F}$.

And hence all possible unions of the A_j's also must $\in \mathcal{F}$ (by A3). On the other hand, they already form a σ-algebra

<u>A1</u>: $\Omega = \bigcup_{j=1}^{4} A_j$, ok; <u>A2</u>: $\overline{\bigcup_{j\in I} A_j} = \bigcup_{j\notin I} A_j$, ok ($I$ is a subset of $\{1,2,3,4\}$)

<u>A3</u>: obvious, as unions of unions of the sets A_j can only be such unions!

- Therefore the coll'n of all possible unions of the A_j's is the <u>smallest</u> σ-algebra containing A, B:

$$\mathcal{F} = \{\emptyset, A_1, A_2, A_3, A_4, A_1\cup A_2, A_1\cup A_3, A_1\cup A_4, A_2\cup A_3, A_2\cup A_4, A_3\cup A_4, \bar{A}_1, \bar{A}_2, \bar{A}_3, \bar{A}_4, \Omega\} \equiv \{\emptyset, A\cap\bar{B}, B\cap\bar{A}, D, C, (A\cap\bar{B})\cup(\bar{A}\cap B), A, \bar{B}, B, \bar{A}, D\cup C, C\cup B, C\cup A, \bar{D}, A\cup B, \Omega\},$$

$|\mathcal{F}| = 16.$

(b) $X(\omega)$ is a RV on $(\Omega, \mathcal{F}) \Leftrightarrow X(\omega) = \sum_{k=1}^{4} c_k \mathbb{1}_{A_k}$ HW-3

for some numbers c_k, $k = 1, \ldots, 4$.

• Clearly, such an X satisfies the def^n of a RV:

$$\{X \in B\} = \bigcup_{k : c_k \in B} \underbrace{\{X = c_k\}}_{A_k} \in \mathcal{F}$$

(if all c_j's are different; otherwise can be a union of some A_j's - s.t. $c_j = c_k$)

• If X is not of this form, then, for some k, it will assume at least two different values on A_k: $\exists \omega_1, \omega_2 \in A_k$ s.t. $x_1 = X(\omega_1) \neq X(\omega_2) = x_2$

But this means that the set

$$A_k' := \underbrace{\{X = x_2\}}_{\substack{\cap \\ \mathcal{F}}} \cap \underbrace{A_k}_{\substack{\cap \\ \mathcal{F}}} \in \mathcal{F}$$

since X is a RV

by cond'n

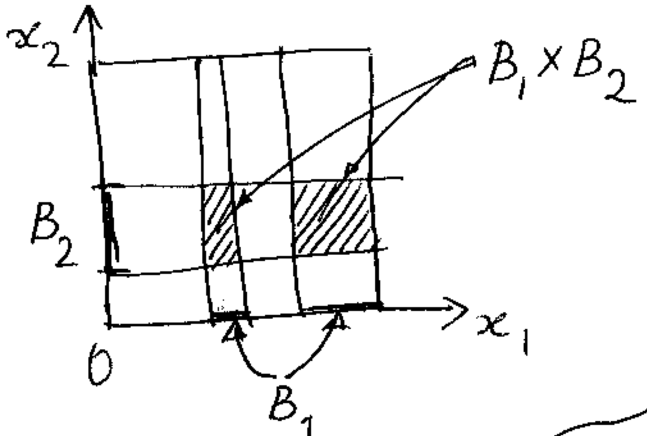
by properties of $\mathcal{F}$

is strictly smaller than A_k (as $\omega_1 \in A_k$ but $\omega_1 \notin A_k'$: $X(\omega_1) = x_1 \neq x_2$) — but there is no such set in $\mathcal{F}$, so - contradiction!

2 (a) Independence of $X_1, X_2 \Leftrightarrow \forall B_1, B_2 \in \mathcal{B}$,

$$\underbrace{P(X_1 \in B_1, X_2 \in B_2)}_{\parallel} \overset{?}{=} \underbrace{P(X_1 \in B_1)}_{= \text{length of } B_1} \underbrace{P(X_2 \in B_2)}_{= \text{length of } B_2}$$ $\boxed{\text{SO OK}}$

$P((X_1, X_2) \in B_1 \times B_2) =$ area of the "rectangle" $B_1 \times B_2 \equiv \{(x_1, x_2) : x_1 \in B_1, x_2 \in B_2\}$



Or, equivalently, need to show that for $\forall x_1, x_2$ we have the foll'g equality for the distr^n functions:

$$\underbrace{F_{X_1,X_2}(x_1,x_2)}_{\parallel} \stackrel{?}{=} F_{X_1}(x_1)\,F_{X_2}(x_2) = x_1 x_2, \quad x_i \in [0,1]$$

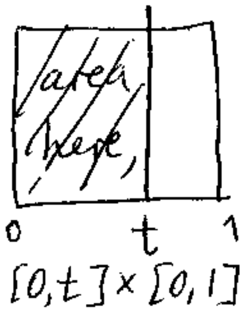
as $F_{X_i}(t) = t$, $t \in [0,1]$, $i = 1,2$.

$P(X_1 \le x_1, X_2 \le x_2) =$

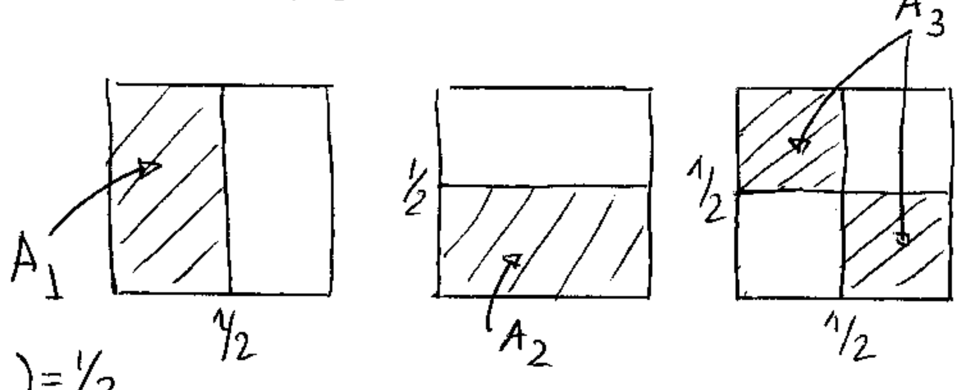
$=$ area of $[0,x_1] \times [0,x_2] = x_1 x_2$, so OK

In both cases we used the (almost obvious) fact that $X_i \sim U(0,1)$ (uniform on $(0,1)$). How to show:

$$F_{X_i}(t) = P(X_i \le t) = \text{area here} = t, \quad t \in [0,1].$$



(b) Depict the events:



Clearly, $P(A_i) = 1/2$, $i = 1,2,3$; and

$$\begin{cases} A_1 A_2 = [0,1/2]^2, \text{ so } P(A_1 A_2) = 1/4 \\ A_1 A_3 = [0,1/2] \times [1/2,1], \text{ so } P(A_1 A_3) = 1/4 \\ A_2 A_3 = [1/2,1] \times [0,1/2], \text{ so } P(A_2 A_3) = 1/2. \end{cases}$$

In all 3 cases $P(A_i A_j) = P(A_i)P(A_j)$, <u>Bingo</u>.

(c) Clearly, $A_1 A_2 A_3 = \emptyset$, so

$$0 = P(A_1 A_2 A_3) \neq P(A_1)P(A_2)P(A_3) = 1/8.$$

(d) <u>No</u>, since $(A_1 A_2) \cap A_3 = \emptyset$ (cf. (c)), i.e. the events are <u>disjoint</u>, whereas both $A_1 A_2$ and A_3 have positive probs:

$$0 = P((A_1 A_2) \cap A_3) \neq \underbrace{P(A_1 A_2)}_{= 1/4} \underbrace{P(A_3)}_{= 1/2} = 1/8.$$

(cf. (b))