

Homework 4 solutions

620-302

1
HW-4

1. (a) $E(S_{n+m} | S_n) = E(S_n + X_{n+1} + \dots + X_{n+m} | S_n)$

$\stackrel{\text{CE1}}{=} \underbrace{E(S_n | S_n)}_{\substack{\text{CE2} \\ S_n}} + \underbrace{E(X_{n+1} + \dots + X_{n+m} | S_n)}_{\substack{\text{CE3, indep'ce} \\ E(X_{n+1} + \dots + X_{n+m}) = mEX_1 = m\mu}}$

$= S_n + m\mu.$

(b) $E(X_1 | S_n) \stackrel{\text{as all } E(X_j | S_n) = E(X_1 | S_n), j=1, \dots, n, \text{ by symm.}}{=} \frac{1}{n} (E(X_1 | S_n) + \dots + E(X_n | S_n))$
 (the order of the X's in the sum doesn't matter)

$\stackrel{\text{CE1}}{=} \frac{1}{n} E(X_1 + \dots + X_n | S_n) \stackrel{\text{CE2}}{=} \frac{1}{n} S_n.$

(c) $E(S_{n+m}^2 | S_n) = E[(S_n + (X_{n+1} + \dots + X_{n+m}))^2 | S_n]$

$\stackrel{\text{CE1}}{=} E(S_n^2 | S_n) + 2E(S_n(X_{n+1} + \dots + X_{n+m}) | S_n)$
 (CE2)

$+ E((X_{n+1} + \dots + X_{n+m})^2 | S_n)$

$\stackrel{\text{indep'ce, CE3, CE2}}{=} S_n^2 + 2S_n \underbrace{E(X_{n+1} + \dots + X_{n+m})}_{m\mu} + \underbrace{E(X_{n+1} + \dots + X_{n+m})^2}_{\text{Var}(\cdot) + (E(\cdot))^2 = m\sigma^2 + m^2\mu^2}$

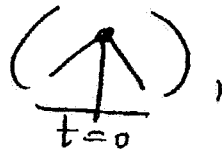
$= S_n^2 + 2S_n m\mu + m\sigma^2 + m^2\mu^2$

$= \underline{\underline{(S_n + m\mu)^2 + m\sigma^2}}$

$$(d) E(S_m | S_n) = E(X_1 + \dots + X_m | S_n)$$

$$\stackrel{\boxed{E-1}}{=} E(X_1 | S_n) + \dots + E(X_m | S_n) \stackrel{\substack{\uparrow \\ \text{by symmetry}}}{=} m E(X_1 | S_n) = \frac{m}{n} S_n.$$

2. (a) • $\varphi(t)$ is real-valued \Rightarrow the distn is symmetric (about 0).

• φ is not differentiable at $t=0$ , so we have $E|X| = \infty$.

• $\int_{-\infty}^{\infty} |\varphi(t)| dt = \int_{-1}^1 |\varphi(t)| dt \leq 2$ (as $|\varphi(t)| \leq 1$),

so the distn has a continuous density.

• Moreover, as also $\int_{-\infty}^{\infty} |t^k \varphi(t)| dt = \int_{-1}^1 |t^k \varphi(t)| dt \leq 2$, $k=1, 2, \dots$, the density will be infinitely many times differentiable.

(b) By inversion f^{la} (sl. (A1)), $f(x) = \frac{1}{2\pi} \int_{-1}^1 e^{-itx} (1-|t|) dt$

$$= \frac{1}{2\pi} \left[\int_{-1}^1 \cos(tx) (1-|t|) dt - i \underbrace{\int_{-1}^1 \sin(tx) (1-|t|) dt}_{=0} \right]$$

since the answer must be a real number (alternatively, note that we integrate an odd function over $[-1, 1] \rightarrow$ zero!)

Now use $|t| = \begin{cases} -t, & t \leq 0, \\ t, & t \geq 0, \end{cases}$ to compute HW-4

$$\int_{-1}^1 (1-|t|) \cos(tx) dx = \int_{-1}^0 \underbrace{(1+t)}_{u_+} \underbrace{\cos(tx)}_{dv} dt + \int_0^1 \underbrace{(1-t)}_{u_-} \underbrace{\cos(tx)}_{dv} dt$$

By parts,

since $du_{\pm} = \pm dt$, $v = \frac{\sin(tx)}{x}$

NB: We assume here that $x \neq 0$; if $x=0$, one can imm. integrate to get $f(0) = \frac{1}{2\pi}$

$$= \underbrace{\left[(1+t) \frac{\sin(tx)}{x} \right]_{-1}^0}_{=0} - \frac{1}{x} \int_{-1}^0 \sin(tx) dt +$$

$$+ \underbrace{\left[(1-t) \frac{\sin(tx)}{x} \right]_0^1}_{=0} + \frac{1}{x} \int_0^1 \sin(tx) dt$$

$$= - \left[-\frac{\cos(tx)}{x^2} \right]_{-1}^0 + \left[-\frac{\cos(tx)}{x^2} \right]_0^1 =$$

$$= \frac{1}{x^2} (1 - \cos x) + \frac{1}{x^2} (-\cos x + 1) = \frac{2(1 - \cos x)}{x^2}$$

$$= \frac{4 \sin^2(x/2)}{x^2}, \text{ so that } f(x) = \frac{2 \sin^2(x/2)}{\pi x^2}.$$

NB: For $x=0$, this formula gives $\frac{1}{2\pi}$ as well.