Homework 4 solutions

1. (a) $E(S_{n+m}|S_n) = E(S_n + X_{n+1} + \dots + X_{n+m}|S_n)$

$$\overset{\boxed{CE1}}{=} \underbrace{E(S_n|S_n)}_{\overset{\boxed{CE2}}{=} S_n} + \underbrace{E(X_{n+1} + \dots + X_{n+m}|S_n)}_{\overset{\boxed{CE3},\ \text{indep'ce}}{=} E(X_{n+1}+\dots+X_{n+m}) = m EX_1 = m\mu}$$

$$= S_n + m\mu.$$

(b) $E(X_1|S_n) \overset{\text{as all } E(X_j|S_n) = E(X_1|S_n),\ j=1,\dots,n,}{=} \frac{1}{n}(E(X_1|S_n) + \dots + E(X_n|S_n))$

By symm. (the order of the X's in the sum doesn't matter)

$$\overset{\boxed{CE1}}{=} \frac{1}{n} E(\underbrace{X_1 + \dots + X_n}_{= S_n}|S_n) \overset{\boxed{CE2}}{=} \frac{1}{n} S_n.$$

(c) $E(S^2_{n+m}|S_n) = E[(S_n + (X_{n+1} + \dots + X_{n+m}))^2|S_n]$

$$\overset{\boxed{CE1}}{=} E(S_n^2|S_n) + 2E(S_n(X_{n+1} + \dots + X_{n+m})|S_n)$$

$\boxed{CE2}$

$$+ E((X_{n+1} + \dots + X_{n+m})^2|S_n)$$

indep'ce, $\boxed{CE3}$ $\boxed{CE2}$

$$= S_n^2 + 2S_n \underbrace{E(X_{n+1} + \dots + X_{n+m})}_{m\mu} + \underbrace{E(X_{n+1} + \dots + X_{n+m})^2}_{= Var(\cdot) + (E(\cdot))^2 = m\sigma^2 + m^2\mu^2}$$

$$= S_n^2 + 2S_n m\mu + m\sigma^2 + m^2\mu^2$$

$$= (S_n + m\mu)^2 + m\sigma^2.$$

(d) $E(S_m \mid S_n) = E(X_1 + \cdots + X_m \mid S_n)$

$$\overset{\boxed{CE\text{-}1}}{=} E(X_1 \mid S_n) + \cdots + E(X_m \mid S_n) \underset{\text{by symmetry}}{=} m E(X_1 \mid S_n) = \frac{m}{n} S_n .$$

2. (a) • $\varphi(t)$ is real-valued $\Rightarrow$ the distr^n is symmetric (about 0).

• φ is not differentiable at $t=0$, so we have $E|X| = \infty$.

• $\int_{-\infty}^{\infty} |\varphi(t)|\,dt = \int_{-1}^{1} |\varphi(t)|\,dt \le 2$ (as $|\varphi(t)| \le 1$), so the distr^n has a continuous density.

• Moreover, as also $\int_{-\infty}^{\infty} |t^k \varphi(t)|\,dt = \int_{-1}^{1} |t^k \varphi(t)|\,dt \le 2$, $k = 1, 2, \ldots$, the density will be infinitely many time differentiable.

(b) By inversion f^{la} (sl. 41), $f(x) = \frac{1}{2\pi} \int_{-1}^{1} e^{-itx} (1 - |t|)\,dt$

$$= \frac{1}{2\pi} \left[\int_{-1}^{1} \cos(tx)(1-|t|)\,dt - i \underbrace{\int_{-1}^{1} \sin(tx)(1-|t|)\,dt}_{=0} \right]$$

since the answer must be a real number (alternatively, note that we integrate an odd function over $[-1, 1] \rightarrow$ zero!)

Now use $|t| = \begin{cases} -t, & t \le 0, \\ t, & t \ge 0, \end{cases}$ to compute

$$\int_{-1}^{1} (1-|t|)\cos(tx)\,dx = \int_{-1}^{0} \underbrace{(1+t)}_{u_+} \underbrace{\cos(tx)\,dt}_{dv} + \int_{0}^{1} \underbrace{(1-t)}_{u_-} \underbrace{\cos(tx)\,dt}_{dv}$$

by parts, since $du_\pm = \pm dt$, $v = \dfrac{\sin(tx)}{x}$

NB: We assume here that $x \neq 0$; if $x=0$, one can imm. integrate to get $f(0) = \frac{1}{2\pi}$

$$= \underbrace{\left[(1+t)\frac{\sin(tx)}{x}\right]_{-1}^{0}}_{=0} - \frac{1}{x}\int_{-1}^{0} \sin(tx)\,dt +$$

$$+ \underbrace{\left[(1-t)\frac{\sin(tx)}{x}\right]_{0}^{1}}_{=0} + \frac{1}{x}\int_{0}^{1} \sin(tx)\,dt$$

$$= -\left[-\frac{\cos(tx)}{x^2}\right]_{-1}^{0} + \left[-\frac{\cos(tx)}{x^2}\right]_{0}^{1} =$$

$$= \frac{1}{x^2}(1-\cos x) + \frac{1}{x^2}(-\cos x + 1) = \frac{2(1-\cos x)}{x^2}$$

$$= \frac{4\sin^2(x/2)}{x^2}, \text{ so that } f(x) = \frac{2\sin^2(x/2)}{\pi x^2}.$$

NB: For $x=0$, this formula gives $\frac{1}{2\pi}$ as well.