Homework 5 solutions

1. (a) $$\varphi(t) = Ee^{itZ^2} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{itx^2} e^{-x^2/2}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x^2(1-2it)/2}\,dx$$

$$= \frac{1}{\sqrt{1-2it}}\cdot\underbrace{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(ax)^2/2}\,d(ax)}_{=1 \text{ (as if } a \text{ were a real number)}} = (1-2it)^{-1/2}$$

with $a = \sqrt{1-2it}$

(b) $$EZ^2 \overset{(6),\,(137)}{=} -i\varphi'(0) = -i \times \frac{-(-2i)}{2(1-2it)^{3/2}}\Big|_{t=0}$$

$$= -i(1-2it)^{-3/2}\Big|_{t=0} = 1;$$

$$EZ^4 = -\varphi''(0) = -i\times\left(-\frac{3}{2}\right)\times(-2i)(1-2it)^{-5/2}\Big|_{t=0} = 3.$$

(c) $$\varphi_{\chi^2_m}(t) = (\varphi(t))^m = (1-2it)^{-m/2}$$

as $\chi^2_m = Z_1^2 + \cdots + Z_m^2$

(d) Not stable as e.g.

$$\varphi_{\chi^2_m}(t)^2 = \varphi_{\chi^2_m}(t)\,\varphi_{\chi^2_m}(t) \neq e^{itb}\varphi_{\chi^2_m}(at) \quad \text{for some } a,b$$

But ID: for any $n = 1, 2, \ldots$,

$$\left(\varphi_{\chi^2_m}(t)\right)^{1/n} = (1-2it)^{-m/2n} \quad \text{is the ch.f.}$$

of $\Gamma(m/2n, 1/2)$ distr^n (cf. tutorial prob. 1)

2. (a) $\varphi_n(t) = Ee^{itY_1^{(n)}} = \frac{1}{n}e^{it\cdot 1} + \left(1-\frac{4}{n}\right)e^{it\cdot 0} +$

$$+\frac{3}{n}e^{it\cdot(-1)} = 1 + \frac{1}{n}\left(e^{it} + 3e^{-it} - 4\right).$$

(b) $\varphi_{X_n}(t) = \varphi_n^n(t) = \left(1 + \frac{1}{n}\left(e^{it} + 3e^{-it} - 4\right)\right)^n$

$$= \left[e^{\frac{1}{n}(e^{it}+3e^{-it}-4)+o(n^{-1})}\right]^n = e^{e^{it}+3e^{-it}-4+o(1)}$$

$$\xrightarrow[n\to\infty]{} e^{e^{it}+3e^{-it}-4},$$ cont's at 0, hence the ch.f. of Y

(c) $$= e^{e^{it}-1}\,e^{3(e^{-it}-1)} = \underset{Po(1)}{\text{ch.f. of}} \times \underset{Po(3)}{\text{ch.f. of}}$$ (sl. 153)

i.e. the ch.f. of the sum of 2 indep't Poisson RV's with parameters 1 and 3, resp.

(c) $= e^{e^{it}-1} e^{3(e^{-it}-1)} = \varphi_1(t)\varphi_3(-t)$, $\varphi_\lambda(t) = $ ch.f. of Po(λ)

Since $\varphi_X(-t) = \varphi_{-X}(t)$, we got the product of two ch.f.'s, one of Po(1)-RV and the other of $-$(Po(3)-RV), that is the ch.f. of the sum of two indep't RV's with these distr's: $N_1 - N_2$, where $N_2 \sim$ Po(3).