# Homework-6 (solutions)

1. (a) $\{\tau_1 \vee \tau_2 \le t\} = \underbrace{\{\tau_1 \le t\}}_{\in \mathcal{F}_t} \cap \underbrace{\{\tau_2 \le t\}}_{\in \mathcal{F}_t} \in \mathcal{F}_t$, so OK, ST

(b) No: $\{\tau = t\} = \left[\bigcap_{k=1}^{t} \{X_k > 0\}\right] \cap \underbrace{\{X_{t+1} \le 0\}}_{\notin \mathcal{F}_t \text{ in general!}}$

(c) Yes: $\{\tau \le t\} = \{\tau_1 + 2 \le t\} = \{\tau_1 \le t-2\} \in \mathcal{F}_{t-2} \subset \mathcal{F}_t$.

(d) Yes: $\{\tau = t\} = \{\tau_1 + \tau_2 = t\} =$

$$= \bigcup_{k=0}^{t} \Big[\underbrace{\{\tau_1 = t-k\}}_{\substack{\in \mathcal{F}_{t-k} \in \mathcal{F}_t \\ (k \ge 0)}} \cap \underbrace{\{\tau_2 = k\}}_{\substack{\in \mathcal{F}_k \subset \mathcal{F}_t \\ (k \le t)}}\Big] \in \mathcal{F}_t.$$

2. (a) Clearly, it's integrable: $E|X_n| < \infty$; consider the natural filtration $\{\mathcal{F}_n\}$ (as no other is specified).

$$E(X_{n+1} \mid \mathcal{F}_n) = E\left(\left(\frac{q}{p}\right)^{S_n + Y_{n+1}} \,\Big|\, \mathcal{F}_n\right)$$

(where $Y_{n+1}$ = displacement of the particle on step $n+1$; $Y_{n+1} = \begin{cases} +1 & \text{w.p. } p \\ -1 & \text{w.p. } q \end{cases}$; and indep't of $\mathcal{F}_n$)

$$= \left(\frac{q}{p}\right)^{S_n} E\left(\left(\frac{q}{p}\right)^{Y_{n+1}} \Big| \mathcal{F}_n\right) =$$

as $S_n$ is $\mathcal{F}_n$-measurable

since $Y_{n+1}$ and $\mathcal{F}_n$ are independent

$$= X_n \, E\left(\frac{q}{p}\right)^{Y_{n+1}} = X_n \left[\left(\frac{q}{p}\right)^1 \times p + \left(\frac{q}{p}\right)^{-1} \times q\right] = X_n.$$

(b) $1 = EX_0 = EX_\tau = r^b \underbrace{P(X_\tau = b)}_{=:p_+} + r^a \underbrace{P(X_\tau = a)}_{=1-p_+}$

putting $r = q/p$

$$= r^b p_+ + r^a(1-p_+) = (r^b - r^a)\, p_+ + r^a,$$

so that

$$P(X_\tau = b) = p_+ = \frac{1-r^a}{r^b - r^a} \; ; \; P(X_\tau = a) = 1 - p_+ = \frac{r^b - 1}{r^b - r^a}.$$

Now OST for $\{Z_n\}, \tau$:

$$0 = EZ_0 = EZ_\tau = E(S_\tau - (p-q)\tau)$$
$$= ES_\tau - (2p-1)E\tau = b p_+ + a(1-p_+) - (2p-1)E\tau,$$

so that

$$E\tau = \frac{b(1-r^a) + a(r^b - 1)}{(2p-1)(r^b - r^a)}$$

BTW: note what happens when $p \to 1/2$ (i.e. $r \to 1$).