Solutions to HW-7

1. Using independence of increments:

$$X \equiv \underbrace{W_0}_{=0} + W_2 - W_3 + 2W_4 = W_2 - W_3 + 2(W_4 - W_3) + 2W_3$$

$$= \underbrace{2W_2}_{\sim 2\times N(0,2) \,\sim N(0,8)} + \underbrace{(W_3 - W_2)}_{\sim N(0,1)} + \underbrace{2(W_4 - W_3)}_{\sim 2\times N(0,1)\,\sim N(0,4)}$$

$$\sim N(0, 8+1+4) = N(0,13)$$

(by indep'ce)

2. (a) As $\{X_t\}$ is obtained from $\{W_t\}$ simply by time-scaling (and $\times a$), it must also have independent increments & contin's trajectories. So we only need to verify that $X_t - X_s \sim N(0, t-s)$, $s<t$, which is obvious: $X_t - X_s = a(W_{t/a^2} - W_{s/a^2}) \sim N(0, a^2 \times (t/a^2 - s/a^2)) = N(0, t-s)$.

[An alternative approach: note that $\{X_t\}$ is still a Gaussian process & compute its mean & covariance functions.]

BTW: this property is referret to as "self-similarity" of the BM process.

(b) Since, due to symmetry, $\{W_t\} \overset{d}{=} \{-W_t\}$ we have $W_{t_1} W_{t_2} W_{t_3} \overset{d}{=} (-W_{t_1})(-W_{t_2})(-W_{t_3})$, and so

$$\mathbb{E}(W_{t_1} W_{t_2} W_{t_3}) = \mathbb{E}(-W_{t_1} W_{t_2} W_{t_3}) = -\mathbb{E}(W_{t_1} W_{t_2} W_{t_3}),$$

which is only possible is this is zero.

3. (a) Using the MG $\{W_t\}$ & OST:

$$0 = EW_0 \overset{OST}{=} E\underbrace{W_\tau}_{=2\tau-4} = E(2\tau - 4) = 2E\tau - 4 \Rightarrow E\tau = 2$$

(as $W_0 = 0$; $=2\tau-4$, the value u_τ (we have $W_\tau = u_\tau$ since $\{W_t\}$ is cont's))

(b) Using the MG $X_t = W_t^2 - t$:

$$0 = EX_0 \overset{OST}{=} EX_\tau = E(W_\tau^2 - \tau) =$$

(as $X_0 = 0$; as $W_\tau = u_\tau = 2\tau - 4$)

$$= E((2\tau-4)^2 - \tau) = E(4\tau^2 - 17\tau + 16)$$

$$= 4E\tau^2 - 17E\tau + 16 \underset{(a)}{=} 4E\tau^2 - 17\times 2 + 16$$

$$= 4E\tau^2 - 18 \Rightarrow E\tau^2 = 4.5, \text{ and}$$

hence $Var(\tau) = E\tau^2 - (\underbrace{E\tau}_{\overset{(a)}{=}2})^2 = 4.5 - 4 = \frac{1}{2}$

(c) Using the MG $Y_t = e^{vW_t - v^2 t/2}$:

$$1 = EY_0 \overset{OST}{=} EY_\tau = E e^{vW_\tau - v^2\tau/2}$$

(as $Y_0 = 1$; $W_\tau = 2\tau - 4$, as above)

$$= E\exp\{-4v - (v^2/2 - 2v)\tau\}$$

$$= e^{-4v} E e^{-s\tau} \Rightarrow E e^{-s\tau} \overset{(*)}{=} e^{4v},$$

putting $s := v^2/2 - 2v$ — solve this for v:

$v = 2 \pm \sqrt{2(s+2)}$, choose "$-$" (since $s = 0 \Rightarrow v = 0$, cf. (*)), and hence

$$l_\tau(s) = \exp\left(8 - 4\sqrt{2(s+2)}\right)$$

(d) $EW_\tau \overset{OST}{=} EW_0 = 0$; from (b),

$0 = E(W_\tau^2 - \tau) = EW_\tau^2 - E\tau$, so that

$$EW_\tau^2 = E\tau = 2.$$