Solutions to HW-8

1. (a) By the formula for conditional dens's: $f_{Y|X}(y|x) = \dfrac{f_{(X,Y)}(x,y)}{f_X(x)}$,

$$f_{(2W_3,W_5)}(x,y) = f_{W_5|2W_3}(y|x)\, \underbrace{f_{2W_3}(x)}_{\sim N(0,2^2\times 3)}$$

$$\boxed{W_5 = \underbrace{W_5 - W_3 + x/2}_{\sim N(x/2,\,2)} \text{ if } 2W_3 = x}$$

$$= \frac{1}{\sqrt{2\pi\cdot 2}} e^{-(y-x/2)^2/4} \times \frac{1}{\sqrt{2\pi\times 12}} e^{-x^2/24}$$

$$= \frac{1}{4\pi\sqrt{6}} \exp\left\{-\frac{1}{4}y^2 + \frac{1}{4}xy - \frac{5}{48}x^2\right\}.$$

(b) Again by the formula for cond'l densities (with $X=(W_2, 2W_3)$, $Y = W_5$)

$$f_{(W_2,2W_3,W_5)}(x,y,z) =$$

$$= f_{W_5|(W_2,2W_3)}(z|(x,y))\, \underbrace{f_{(W_2,2W_3)}(x,y)}_{\text{found in Tute 8, Pr. 2}}$$

$$\boxed{\begin{array}{l} W_5 - W_3 + y/2 \\ \text{if } (W_2, 2W_3) = (x,y), \\ \sim N(y/2,\, 2) \end{array}}$$

$$= \frac{e^{-(z-y/2)^2/4}}{\sqrt{2\pi\cdot 2}} \times \frac{e^{-\frac{1}{8}y^2 + \frac{1}{2}xy - \frac{3}{4}x^2}}{4\pi\sqrt{2}}$$

$$= \frac{1}{8\pi\sqrt{2\pi}} \exp\left\{-\frac{3}{4}x^2 + \frac{1}{2}xy - \frac{3}{16}y^2 + \frac{1}{4}yz - \frac{1}{4}z^2\right\}$$

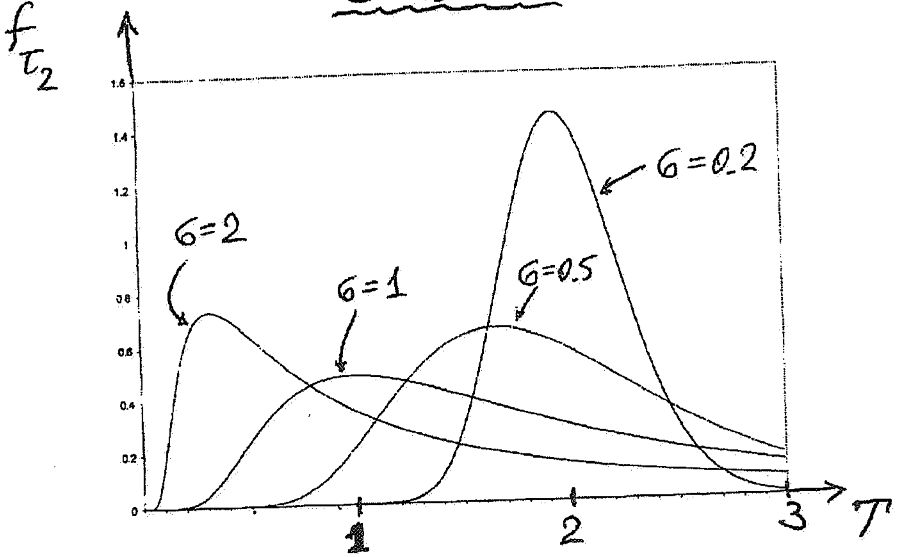
2. As $X_T = T + \sigma W_T \sim N(T, \sigma^2 T)$, we get

$$f_{\tau_x}(T) = \frac{x}{T} f_{X_T}(x) = \frac{x}{T} \times \frac{e^{-(x-T)^2/2\sigma^2 T}}{\sqrt{2\pi}\,\sigma\sqrt{T}}$$

So for $x=2$:

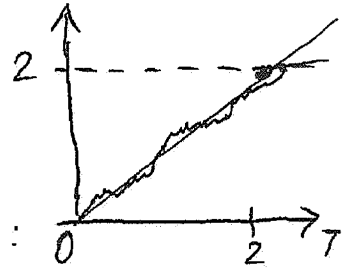
$$f_{\tau_2}(T) = \frac{\sqrt{2}}{\sqrt{\pi}\,\sigma\,T^{3/2}}\, e^{-(T-2)^2/2\sigma^2 T}, \quad T>0.$$

<u>Densities</u> for diff^t σ's:



- For small σ, the density f_{τ_2} is "bell-shaped", unimodal, nearly symmetric (about $T=2$), with a small spread.
  As σ increases, the mode of the density moves to the left, the spread increases, and the shape becomes more and more skew to the right.

- When σ is small, the trajectory of $\{X_t\}$ closely follows the mean f^n $EX_t = t$:



So it hits the level $x=2$ in a vicinity of the point $T=2$.

For larger σ, random oscillations about the mean f^n become more profound, increasing the prob'ty of hitting $x=2$ very soon (or, in case $\{X_t\}$ first <u>droops</u> far enough pretty late).