Solutions to HW-9

620302

1. (a) For $f(x)=e^{-2x}$, $f'(x)=-2e^{-2x}=-2f(x)$, $f''(x)=4f(x)$, so by Itô's formula

$$df(X_t)\overset{265}{=}f'(X_t)\,dX_t+\tfrac{1}{2}f''(X_t)(dX_t)^2$$

$$=-2f(X_t)(dt+dW_t)+\tfrac{1}{2}4f(X_t)\,dt \quad \text{as } (dX_t)^2=(dW_t)^2=dt$$

$$=-2e^{-2X_t}\,dW_t \quad \text{or} \quad =-2e^{-2(t+W_t)}\,dW_t.$$

(b) Yes, since $Y_t=\underbrace{Y_0}_{=\exp\{-2X_0\}=1}+\int_0^t dY_s$

$$=1-2\underbrace{\int_0^t e^{-2s-2W_s}\,dW_s}_{=I_t(g),\ g_s=e^{-2s-2W_s}},$$

II is an MG [$\boxed{II3}$ on slide 258], and hence $1-2I_t(g)$ is also an MG:

$$E(1-2I_t(g)\,|\,\mathcal{F}_s)=1-2E(I_t(g)\,|\,\mathcal{F}_s)$$

$$\underset{\text{II is an MG}}{=}1-2I_s(g),\quad 0\le s\le t.$$

2 (a) For $f(x)=\sqrt{x}$, $f'(x)=\frac{1}{2\sqrt{x}}$, $f''(x)=\frac{-1}{4x^{3/2}}$, so by Itô's f'la,

$$dZ_t=df(V_t)=f'(V_t)\,dV_t+\tfrac{1}{2}f''(V_t)\underbrace{(dV_t)^2}_{=4V_t\,dt}$$

$$=\frac{1}{2\sqrt{V_t}}\left((1-V_t)dt+2\sqrt{V_t}\,dW_t\right)+\frac{1}{2}\left(\frac{-1}{4V_t^{3/2}}\right)\cdot 4V_t\,dt$$

$$=-\tfrac{1}{2}\sqrt{V_t}\,dt+dW_t=-\tfrac{1}{2}Z_t\,dt+dW_t=dZ_t,$$

$Z_0=\sqrt{V_0}=1$.

(b) $Z_t=X_tY_t$, with $X_t=e^{-t/2}$, $Y_t=1+\int_0^t e^{s/2}\,dW_s$, so by Itô's product rule

$$dZ_t=\underbrace{dX_t}_{=-\frac{1}{2}e^{-t/2}dt=-\frac{1}{2}X_t\,dt}\cdot Y_t+X_t\cdot dY_t+\underbrace{dX_t\cdot dY_t}_{=0,\text{ as } X_t \text{ is non-random}}$$

$$=-\tfrac{1}{2}\underbrace{X_tY_t}_{=Z_t}\,dt+\underbrace{X_te^{t/2}}_{=1}\,dW_t+0$$

$$=-\tfrac{1}{2}Z_t\,dt+dW_t,\ \text{OK!}$$

Initial value: $Z_0=e^{-0}\left(1+\int_0^0\right)=1$, OK!