Solution to HW-10.

1. (a) As $\mu(x) = 1 - x$, $\sigma(x) = \sqrt{2x}$ (so that $\sigma^2/2 = x$), $x > 0$, we get:

BWKE: for $v = v(s,x)$,

$$v'_s = (x-1)v'_x - x v''_{xx}.$$

FWKE: for $u = u(t,y)$,

$$u'_t = -((1-y)u)'_y + (yu)''_{yy}$$

$$= u + (y-1)u'_y + 2u'_y + y u''_{yy}$$

$$= u + (y+1)u'_y + y u''_{yy}.$$

(b) Eq'n for the stationary density $\pi = \pi(y)$: from FWKE & "zero rule" (see slide 306a), we get

$$(y-1)\pi + \underbrace{(y\pi)'}_{\text{"}y\pi' + \pi} = 0,$$

so the eq'n is equivalent to:

$y\pi + y\pi' = 0$. Since $y > 0$, the latter is equiv't to: $\pi + \pi' = 0$, i.e.

$(\ln \pi)' = \frac{\pi'}{\pi} = -1$, so that

$\ln \pi = C - y$, and $\pi(y) = C_1 e^{-y}$, $y > 0$.

As $\pi(y)$ is a prob'ty density, must have $\int_0^\infty \pi(y)\,dy = 1$, and hence $C_1 = 1$.

Answer: $\pi(y) = e^{-y}$, $y > 0$

(std. exponential density)

(c) ODE: $0 = (x-1)V' - xV''$; $V(a) = 0$, $V(b) = 1$.

Verifying: $V' = C_1 Ei'(x) = \frac{c_1}{x} + c_1 \sum_{n=1}^{\infty} \frac{\cancel{n} x^{n-1}}{\cancel{n} \cdot n!} = \frac{c_1}{x} \sum_{n=0}^{\infty} \frac{x^n}{n!} = \frac{e^x c_1}{x}$,

$V'' = c_1 \frac{x-1}{x^2} e^x$, so-yes! Using boundary cond's:

$$V(x) = \frac{Ei(x) - Ei(a)}{Ei(b) - Ei(a)}, \quad a \le x \le b.$$