HOMEWORK 11 : SOLUTION.

1. (a) As $\mu(s,x) = x(1-x)$, $\sigma^2(s,x) = x(1-x)$, the BWKE for $v = v(s,x)$ is:

$$v'_s = -x(1-x)v'_x - \frac{x(1-x)}{2} v''_{xx}$$
$$= x(x-1)\left(v'_x + \frac{1}{2}v''_{xx}\right).$$

FWKE: for $u = u(t,y)$,

$$u'_t = -\left(y(1-y)u\right)'_y + \frac{1}{2}\left(y(1-y)u\right)''_{yy}$$
$$= 2(y-1)u + (1-3y+y^2)u'_y + \frac{1}{2}y(1-y)u''_{yy}.$$

(b) Put $\tau = \min\{t \geq 0 : X_t = 0 \text{ or } X_t = 1\}$, and set the boundary function

$\psi(t,0) = 0$, $\psi(t,1) = 1$, $t \geq 0$. Then the desired probability

$$v(s,x) = E\left(\psi(\tau, X_\tau) \mid X_s = x\right) = P(X_\tau = 1 \mid X_s = x)$$
$$= P(X_\tau = 1 \mid X_0 = x) = V(x) \text{ solves the}$$

BWKE:

$$0 = v'_s = -\mu v'_x - \frac{\sigma^2}{2} v''_{xx} = -\mu V' - \frac{\sigma^2}{2} V''$$

$$= -x(1-x)V' - \frac{1}{2}x(1-x)V'', \quad V(0)=0,\ V(1)=1.$$

For $x \in (0,1)$ this is equivalent to

$0 = V' + \frac{1}{2}V''$, a linear diff'l eq'n with constant coeff's. To solve, begin with the characteristic eq'n: $\frac{1}{2}z^2 + z = 0$, having the roots $z_1 = 0$, $z_2 = -2$, hence the general solution $V(x) = C_1 + C_2 e^{-2x}$.

Boundary cond'ns: $V(0) = C_1 + C_2 = 0$,
$V(1) = C_1 + C_2 e^{-2} = 1$,

solve for $C_1, C_2 \rightarrow$

$$V(x) = \frac{1 - e^{-2x}}{1 - e^{-2}}, \quad x \in (0,1).$$