Solutions to the *homework problems* are to be left in the 620-302 assignment box (**#181**) on the ground floor in the Richard Berry Building (north entrance). **Don't forget** to print your name, student ID, the subject name and code and your lecturer's name (K. Borovkov) on the first page of your solutions! All homework problems should be attempted. Only one (randomly chosen) of them will be marked. All material handed in must be on A4 size paper. Material on different sized paper will not be marked. The form and neatness of work can be considered in marking. Working and/or reasoning **must** be given to obtain full credit. The submission deadline is **5pm on Monday, 24 August 2009**.

Tutorial Problems

1. Let $\Omega = [0,1]$, $\mathcal{F}$ be a σ-algebra such that $A_n = [0, 1/n) \in \mathcal{F}$ for any $n = 1, 2, \ldots$. Show that
   (a) the singleton (= one-point set) $\{1\} \in \mathcal{F}$;
   (b) the half-open interval $[1/4, 1/2) \in \mathcal{F}$;
   (c) $\{0\} \in \mathcal{F}$.
   (d) Is it true that $\{1/3\} \in \mathcal{F}$?

2. Let Ω be a set. In each of the four cases below, describe the class of *all possible* random variables on $(\Omega, \mathcal{F})$ when we take the class $\mathcal{F}$ of events to be:
   (a) $\mathcal{F} = \{\emptyset, \Omega\}$ (the "trivial σ-algebra");
   (b) $\mathcal{F} = 2^\Omega$ (the collection of all subsets of Ω);
   (c) $\mathcal{F}$ generated by a subset $A \subset \Omega$ (i.e. $\mathcal{F} = \{\emptyset, A, \overline{A}, \Omega\}$, the smallest σ-algebra containing A);
   (d) $\mathcal{F}$ generated by subsets $A, B \subset \Omega$, such that $A \cap B = \emptyset$ and $C = \overline{A \cup B} \neq \emptyset$ (i.e. $\overline{A} \neq B$). First draw a Venn diagram and list all elements of $\mathcal{F}$.

3. Let events A and B be independent, $0 < \mathbf{P}(AB) < 1$.
   (a) Are the events A and $\overline{B}$ independent?
   (b) Are the events $\overline{A}$ and $\overline{B}$ independent?
   (c) Are the events A and $\overline{A}$ independent?

4. Let $\Omega = [-3, 3]$ and $\mathbf{P}$ be the triangular distribution with density
$$f(x) = \frac{2 - |x|}{4}, \quad -2 < x < 2.$$
   Put $X(\omega) = k$ if $k \le |\omega| < k + 1$, $k = 0, 1, \ldots$; $\omega \in \Omega$.

(a) Describe $\sigma(X)$ (the σ-algebra generated by X).

(b) Find the distribution P_X of the random variable X.

Homework Problems

1. Let Ω be a set, $A, B \subset \Omega$ its subsets. Suppose that $D = A \cap B \neq \emptyset$ and $C = \overline{A \cup B} \neq \emptyset$.

   (a) Describe the σ-algebra $\mathcal{F}$ generated by the sets A and B. First draw a Venn diagram and then list all the elements of $\mathcal{F}$. How many elements are there in $\mathcal{F}$?

   (b) Describe all possible random variables on $(\Omega, \mathcal{F})$.

2. Let (X_1, X_2) be a random point uniformly distributed in the unit square $\{(x, y) \in \mathbf{R}^2 : 0 \leq x, y \leq 1\}$. Set

   $$A_1 = \{X_1 \leq 1/2\}, \quad A_2 = \{X_2 \leq 1/2\}, \quad A_3 = \{(X_1 - 1/2)(X_2 - 1/2) < 0\}.$$

   (a) Are the random variables X_1 and X_2 independent?

   (b) Show that any two of the events A_1, A_2, A_3 are independent.

   (c) Show that the events A_1, A_2, A_3 are **not** *jointly independent.*

   (d) Are the events $A_1 A_2 \equiv A_1 \cap A_2$ and A_3 independent?