Solutions to the *homework problems* are to be left in the 620-302 assignment box (#**181**) on the ground floor in the Richard Berry Building (north entrance). **Don't forget** to print your name, student ID, the subject name and code and your lecturer's name (K. Borovkov) on the first page of your solutions! All homework problems should be attempted. Only one (randomly chosen) of them will be marked. All material handed in must be on A4 size paper. Material on different sized paper will not be marked. The form and neatness of work can be considered in marking. Working and/or reasoning **must** be given to obtain full credit. The submission deadline is **5pm on Monday, 31 August 2009**.

Tutorial Problems

1. Let X and Y be random variables (RVs), $\mathbf{E}\,X^2 < \infty$. We showed in class that the conditional expectation $\tilde{X} = \mathbf{E}\,(X|Y)$ is the *best in mean quadratic predictor* for X from Y (see the argument on lecture slide 126; just replace $\mathcal{F}_0$ with Y in the condition). Using the properties of conditional expectations, show that the predictor $\tilde{X}$ and the prediction error $X - \tilde{X}$ are *uncorrelated*, i.e. $\mathrm{cov}(\tilde{X}, X - \tilde{X}) = 0$.

2. Let $\{N_t\}_{t\ge 0}$ be a Poisson process with rate $\lambda > 0$, $\mathcal{F}_t = \sigma\{N_s,\ 0 \le s \le t\}$ the "history" of the process up to the time t (formally, $\mathcal{F}_t$ is the σ-algebra generated by all the RVs N_s, $0 \le s \le t$). Using the properties of the Poisson process and conditional expectations, find

   (a) $\mathbf{E}\,(N_{t+s}|\,\mathcal{F}_t)$, $s, t \ge 0$;

   (b) $\mathbf{E}\,(N_{t+s}^2|\,\mathcal{F}_t)$, $s, t \ge 0$;

   (c) $\mathbf{E}\,(N_s|\,\mathcal{F}_t)$ and $\mathbf{E}\,(N_s^2|\,\mathcal{F}_t)$, $0 \le s \le t$;

   (d) $\mathbf{E}\,(N_s|\,N_t)$ and $\mathbf{E}\,(N_s^2|N_t)$, $0 \le s \le t$.

   (e) Compute the characteristic function (ch.f.) $\varphi(u) = \mathbf{E}\,e^{iuN_t}$ of N_t.

3. Suppose that X and Y are independent RVs. It is known that both X and the sum $Z = X + Y$ are Poisson RVs, with parameters $\lambda_X = 2$ and $\lambda_Z = 3.5$, respectively. Using ch.f.'s (or, if you wish, generating functions) and their properties, find the distribution of Y.

4. Let $X_1, X_2, \dots$ be independent identically distributed RVs, $S_n = X_1 + \cdots + X_n$, $S_0 = 0$, and $N \ge 0$ an integer-valued RV, independent of the sequence $\{X_n\}_{n\ge 1}$. Denote by $h(t) = \mathbf{E}\,e^{itX_1}$ the ch.f. of X_1 and by $g(s) = \mathbf{E}\,s^N$ the generating function of N. Using conditional expectations, compute

   (a) $\mathbf{E}\,S_N$ and $\mathbf{E}\,S_N^2$;

(b) the ch.f. $\varphi(t) = \mathbf{E}\, e^{itS_N}$ of S_N.

(c) Find the distribution of S_N when the X_j's are Bernoulli RVs with parameter $r \in (0,1)$ and N is geometric with parameter $p \in (0,1)$:

$$\mathbf{P}(X_j = 1) = 1 - \mathbf{P}(X_j = 0) = r, \quad \mathbf{P}(N = k) = pq^k, \ k = 0, 1, 2, \ldots$$

Hint: Your answer to part (a) should be given in terms of the functions h and g (possibly including their derivatives).

Homework Problems

1. Let $X_1, X_2, \ldots$ be independent identically distributed RVs, $\mathbf{E}\, X_1 = \mu$, $\mathrm{Var}(X_1) = \sigma^2 < \infty$. Put $S_0 = 0$, $S_n = S_{n-1} + X_n$, $n \geq 1$. Compute

   (a) $\mathbf{E}(S_{n+m} | S_n)$, $m, n = 0, 1, 2, \ldots$;

   (b) $\mathbf{E}(X_1 | S_n)$, $n \geq 1$;

   (c) $\mathbf{E}(S_{n+m}^2 | S_n)$, $m, n = 0, 1, 2, \ldots$;

   (d) $\mathbf{E}(S_m | S_n)$, $m = 0, 1, \ldots, n$.

   Hints: (b) $\mathbf{E}(X_1 | S_n) = \mathbf{E}(X_2 | S_n)$ $(n \geq 2)$ etc. by symmetry. (d) You may wish to use the result of one of the parts (a)–(c) above.

2. One can show that

$$\varphi(t) = \begin{cases} 1 - |t|, & -1 \leq t \leq 1, \\ 0 & \text{otherwise}, \end{cases}$$

   is a ch.f. of some distribution.

   (a) Without doing any calculations, what could you say about the distribution (just from the shape of the ch.f.)?

   (b) Compute the density of the distribution.