

Solutions to the *homework problems* are to be left in the 620-302 assignment box (#181) on the ground floor in the Richard Berry Building (north entrance). **Don't forget** to print your name, student ID, the subject name and code and your lecturer's name (K. Borovkov) on the first page of your solutions! All homework problems should be attempted. Only one (randomly chosen) of them will be marked. All material handed in must be on A4 size paper. Material on different sized paper will not be marked. The form and neatness of work can be considered in marking. Working and/or reasoning **must** be given to obtain full credit. The submission deadline is **5pm on Monday, 7 September 2009**.

Tutorial Problems

- Let X be a $\gamma(r, 1)$ -distributed random variable (RV, gamma with parameters $r > 0$ and 1), so that its density is given by

$$f_X(x) = \frac{1}{\Gamma(r)} x^{r-1} e^{-x}, \quad x > 0 \quad (\Gamma(r) \text{ is the gamma function}).$$

- Compute the characteristic function (ch.f.) of X .
 - Using your answer to part (a), compute $\mathbf{E} X^k$, $k = 1, 2, \dots$
 - Let X_1 and X_2 be independent copies of X , and $Y = X_1 - X_2$. Using the properties of ch.f.'s and your answer to part (a), find the ch.f. of Y . From this, identify the distribution of Y when $r = 1$. [*Hint*: We saw this distribution in one of our lecture examples.]
 - For what values of r can we assert that the distribution of the RV Y from part (d) will have a continuous density?
- Assume that there are $n = 5,000$ policies in a given insurance portfolio. For each of the policies, a claim is made with probability $p = 10^{-3}$, and when it is made, its amount is an exponentially distributed RV with mean $\mu = 2$ (kilodollars). We assume that the policies are independent of each other.

Therefore the aggregate claim amount for the portfolio is given by the sum $X = Y_1 + \dots + Y_n$ of independent random variables Y_j with $\mathbf{P}(Y_j = 0) = 1 - p$, the conditional distribution $\mathbf{P}(Y_j \leq x | Y_j > 0)$ being exponential with mean μ .

- Compute the ch.f. of an individual risk Y_j .
- Compute the ch.f. of the aggregate claim amount X and suggest an appropriate approximation to it.
- What is the probability that the aggregate claim amount is equal to zero?

3. Let $S_n = Y_1 + \cdots + Y_n$, $n \geq 1$, $S_0 = 0$, where Y_j are independent identically distributed RVs with $\mathbf{E} Y_j = 0$, $\text{Var}(Y_j) = \sigma^2 < \infty$. Show that $X_n = S_n^2 - n\sigma^2$, $n \geq 0$, is a martingale (a) with respect to the filtration $\mathcal{F}_n = \sigma(Y_1, \dots, Y_n)$; (b) with respect to the natural filtration $\mathcal{F}'_n = \sigma(X_1, \dots, X_n)$. [*Hint:* (b) Just use the result of (a) and the fact that $\mathcal{F}'_n \subset \mathcal{F}_n$ (why does the last relation hold?).]
4. Denote by $\{N_t\}_{t \geq 0}$ a Poisson process with rate $\lambda > 0$. Show that all three processes (a) $N_t - \lambda t$; (b) $(N_t - \lambda t)^2 - \lambda t$; (c) $\exp\{uN_t - \lambda t(e^u - 1)\}$ (u is a fixed real number) are martingales with respect to the filtration $\mathcal{F}_n = \sigma(N_s, s \leq t)$.

Homework Problems

1. Let Z be a standard normal RV.
 - (a) Compute the ch.f. of Z^2 . [*Hint:* When solving this part, you may treat complex coefficients as if they were real numbers; cf. slide 140.]
 - (b) Use your answer to part (a) to compute $\mathbf{E} Z^2$ and $\mathbf{E} Z^4$.
 - (c) Use your answer to part (a) to find the ch.f. of the χ_m^2 distribution (chi-squared distribution with m degrees of freedom, i.e. the distribution of the sum $Z_1^2 + \cdots + Z_m^2$, where the Z_j 's are independent copies of Z).
2. For any $n = 4, 5, 6, \dots$, let

$$Y_j^{(n)} = \begin{cases} +1 & \text{with probability } 1/n, \\ 0 & \text{with probability } 1 - 4/n, \\ -1 & \text{with probability } 3/n, \end{cases} \quad j = 1, 2, \dots, n,$$

be independent RVs.

- (a) Compute the ch.f. of $Y_1^{(n)}$.
- (b) Use your answer to part (a) to show that, as $n \rightarrow \infty$, the RVs $X_n = Y_1^{(n)} + \cdots + Y_n^{(n)}$ converge in distribution to a RV Y . Give the ch.f. of the RV Y .
- (c) Using ch.f.'s, show that $Y \stackrel{d}{=} N_1 - N_2$ (meaning: the LHS has the same distribution as the RHS), where N_j , $j = 1, 2$, are independent Poisson RVs. Find the parameter of N_2 .