Solutions to the *homework problems* are to be left in the 620-302 assignment box (#**181**) on the ground floor in the Richard Berry Building (north entrance). **Don't forget** to print your name, student ID, the subject name and code and your lecturer's name (K. Borovkov) on the first page of your solutions! All homework problems should be attempted. Only one (randomly chosen) of them will be marked. All material handed in must be on A4 size paper. Material on different sized paper will not be marked. The form and neatness of work can be considered in marking. Working and/or reasoning **must** be given to obtain full credit. The submission deadline is **5pm on Monday, 14 September 2009**.

Tutorial Problems

1. Let $\{X_t\}_{t=0,1,\ldots,T}$ be a positive stochastic process adapted to a filtration $\mathbf{F} = \{\mathcal{F}_t\}$. In each of the following cases, say if the RV τ is a stopping time (ST) with respect to (w.r.t.) $\mathbf{F}$ [if the condition in the definition of a random time τ in (c)–(d) is never met for $t \le T$, we just put $\tau = T$ to avoid any inconvenience]. Explain (e.g. expressing events $\{\tau \le t\}$ in terms of the RV's X_k, $k = 1, 2, \ldots$).

   (a) $\tau = m = \text{const}$;

   (b) $\tau = \tau_1 \wedge \tau_2$ $(= \min\{\tau_1, \tau_2\})$, where τ_j are ST's, $j = 1, 2$;

   (c) $\tau = \min\{t \ge 0 : X_{t+1}/X_t > 1\}$;

   (d) $\tau = \min\left\{t \ge 0 : \sum_{k=0}^{t} X_k > X_t^2\right\}$;

   (e) $\tau = \max\{t \le T : X_t > 10\}$.

2. Let $S_0 = 0$ and $S_n = Y_1 + \cdots + Y_n$, $n \ge 1$, where Y_j are i.i.d. RV's with $\mathbf{P}(Y_1 = 1) = 1 - \mathbf{P}(Y_1 = -1) = 1/2$. Denote by $\tau = \min\{n \ge 0 : S_n = a \text{ or } S_n = b\}$ the first time the random walk S_n hits one of the (integer) barriers $a < 0 < b$. Use the Optional sampling theorem (OST, slide 182), without verifying the conditions of the theorem in detail, to:

   (a) find the distribution of S_τ (use the martingale $X_n = S_n$);

   (b) compute $\mathbf{E}\,\tau$ (use the result of (a) and the martingale[1] $X_n = S_n^2 - n\text{Var}(Y_1)$).

3. Suppose you are playing a "fair game" betting \$1 at each play; your fortune after n plays is then given by a process $\{X_n\}_{n\ge 0}$, which is a martingale w.r.t. the "history" $\mathbf{F} = \{\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \ldots\}$.

   Now suppose that for each play $n = 1, 2, \ldots$ your stake can be an *arbitrary bounded amount* Y_n, but you have to decide how much to stake *before* that play, i.e. based on the history up to play $n-1$ (inclusive). Mathematically, this means that, for any n, $|Y_n| \le C_n = \text{const} < \infty$, and $\{Y_n\}_{n\ge 1}$ is a *predictable* (or *previsible*, cf. the definition of trading strategies on slide 58) process, that is, Y_n is $\mathcal{F}_{n-1}$-measurable for all $n = 1, 2, \ldots$

[1] Cf. tutorial problem 3 from PS-5.

In this case, you win the amount $Y_n(X_n - X_{n-1})$ on play n (as it was $X_n - X_{n-1}$ when staking \$1 each time), and hence your total net gain after n plays will be

$$Z_n = \sum_{k=1}^{n} Y_k(X_k - X_{k-1}), \quad n = 1, 2, \ldots; \quad Z_0 = 0.$$

The process $\{X_n\}_{n\geq 0}$ is called the *martingale transform*[2] of X_n by Y_n.

(a) Show that $\{Z_n\}_{n\geq 0}$ is a martingale w.r.t. $\mathbf{F}$.

(b) The betting strategy of doubling when losing is called "martingale". You begin with a unit stake, win each play with probability 1/2 (regardless of the results of all the previous plays), and double your stake for the next play when losing. After each win, you reset the stake size to one. Represent your net gain process $\{Z_n\}$ when using this strategy as a martingale transform: specify the processes $\{X_n\}$ and $\{Y_n\}$.

(c) Under the assumptions of part (b) above, assume that you stop at the time τ of the first win. Find the distribution and expectation of the ST τ and verify if the statement of the OST holds for the process $\{Z_n\}$ and ST τ.

(d) Compute the expectation $\mathbf{E}\,(|Z_n|;\, \tau > n)$ and hence verify if the condition of the OST given on slide 182 is met.

Homework Problems

1. Suppose that $\{X_t\}_{t\geq 0}$ is a stochastic process adapted to a filtration $\mathbf{F} = \{\mathcal{F}_t\}$, and τ_j are ST's, $j = 1, 2$. In each of the following cases, say if the RV τ is a stopping time w.r.t. $\mathbf{F}$. Explain (e.g. expressing events $\{\tau \leq t\}$ in terms of the RV's X_k, $k = 1, 2, \ldots$).

   (a) $\tau = \tau_1 \vee \tau_2$ $(= \max\{\tau_1, \tau_2\})$;

   (b) $\tau = \min\{t \geq 0 : X_{t+1} \leq 0\}$ (in gambling terms, you quit just before getting ruined);

   (c) $\tau = \tau_1 + 2$;

   (d) [an optional question: a bonus mark for a correct solution!] $\tau = \tau_1 + \tau_2$.

2. Let $\{S_n\}_{n\geq 0}$ be a simple random walk: starting at some initial point S_0, the walking particle at each transition goes up 1 with probability $p \in (0, 1)$ and down 1 with probability $q = 1 - p$. Assume that $p \neq 1/2$.

   (a) Show that $X_n = (q/p)^{S_n}$ is a martingale.

   (b) Suppose the walk starts at $S_0 = 0$ and stops at the time $\tau = \min\{n \geq 0 : S_n = a \text{ or } S_n = b\}$, where $a < 0 < b$ are integers. Use the OST and the martingale from part (a) to find the distribution of the RV S_τ, and then the OST and the martingale $Z_n = S_n - n(p - q)$ to compute $\mathbf{E}\,\tau$.

[2] Martingale transforms are discrete analogues of stochastic integrals (to be discussed later in the course) and play an important role in mathematical theory of finance in discrete time.