Solutions to the *homework problems* are to be left in the 620-302 assignment box (#**181**) on the ground floor in the Richard Berry Building (north entrance). **Don't forget** to print your name, student ID, the subject name and code and your lecturer's name (K. Borovkov) on the first page of your solutions! All homework problems should be attempted. Only one (randomly chosen) of them will be marked. All material handed in must be on A4 size paper. Material on different sized paper will not be marked. The form and neatness of work can be considered in marking. Working and/or reasoning **must** be given to obtain full credit. The submission deadline is **5pm on Monday, 5 October 2009** (yes, that's after the break!).

Tutorial Problems

In what follows, $\{W_t\}_{t\ge 0}$ is a standard Brownian motion process.

1. Let $X_0 = 0$ and $X_t = tW_{1/t}$, $t > 0$. Show that $\{X_t\}_{t\ge 0}$ is also a Brownian motion process.

2. Find the distribution of $2W_{t_1} - W_{t_2}$, $0 < t_1 < t_2$.

3. Let $\mu \in \mathbf{R}$, $\sigma, r, S_0 > 0$ be some constants, $S_t = S_0 \exp\{\mu t + \sigma W_t\}$, $t \ge 0$, a geometric Brownian motion process modelling an asset's price dynamics.

   (a) For a fixed value of $\sigma > 0$, find the value of μ such that the discounted price process $X_t = e^{-rt} S_t$, $t \ge 0$, is a martingale.

   (b) Using the value of μ you found in part (a), compute $\mathbf{E}\,(S_T - K)^+$, where $T, K > 0$ are some fixed numbers. Compare the answer with the formula on lecture slide 79.

   Hint: (b) Note that, similarly to the computation on slide 74, the expectation is equal to $\mathbf{E}\,(S_T - K; S_T > K) = \mathbf{E}\,(S_T; S_T > K) - \mathbf{E}\,(K; S_T > K)$, where the expectations on the RHS are not difficult to compute [observe that, say, for a random variable $Z \sim N(0,1)$, to compute the expectation

   $$\mathbf{E}\,(e^Z; Z > C) = \frac{1}{\sqrt{2\pi}} \int_C^\infty e^{x - x^2/2} dx,$$

   you just have to complete the square in the exponential function and recognize the result as just an integral of (another) normal density, cf. slides 111–112].

4. Let $\tau = \min\{t > 0 : W_t = \pm\sqrt{a + bt}\}$ be the first time the Brownian motion crosses one of the two parabolic boundaries $u_t = \sqrt{a + bt}$ and $v_t = -\sqrt{a + bt}$, $t \ge 0$, where $a > 0$ and $b \in (0,1)$ are some constants. Use the OST (slide 182; don't verify the theorem's conditions!) to compute the expectation $\mathbf{E}\,W_\tau$ and hence find $\mathbf{E}\,\tau$.

Homework Problems

1. Find the distribution of $W_0 + W_2 - W_3 + 2W_4$.

2. Let $a \neq 0$ be a real number.

   (a) Show that $X_t = aW_{t/a^2}$ is also a Brownian motion process.

   (b) For $0 < t_1 < t_2 < t_3$, compute $\mathbf{E}\,(W_{t_1} W_{t_2} W_{t_3})$ (either directly or using the result of part (a) above).

3. Denote by $\tau = \min\{t > 0 : W_t \leq 2t - 4\}$ the first time the Brownian motion process crosses the boundary $u_t = 2t - 4$, $t \geq 0$. Using the three martingales of the Brownian motion and the OST (lecture slide 182; don't verify the conditions of the theorem), compute for the stopping time τ its:

   (a) mean value $\mathbf{E}\,\tau$;

   (b) variance $\mathrm{Var}\,(\tau)$;

   (c) Laplace transform $l_\tau(s) = \mathbf{E}\, e^{-s\tau}$, $s \geq 0$.

   (d) Compute $\mathbf{E}\, W_\tau$ and $\mathbf{E}\, W_\tau^2$.