Tutorial Problems

In what follows, \( \{W_t\}_{t \geq 0} \) is a standard Brownian motion process.

1. Compute the mean function \( m_X(t) = \mathbb{E} X_t \) and the covariance function \( \gamma_X(s, t) = \text{Cov}(X_s, X_t) \) when the process \( \{X_t\}_{t \geq 0} \) is:
   
   (a) the arithmetic Brownian motion \( X_t = \mu t + \sigma W_t \);
   
   (b) the geometric Brownian motion \( X_t = e^{\mu t + \sigma W_t} \).

   In both cases, \( \mu \) and \( \sigma \) are some real constants.

2. Compute the joint density of the random vector \( (W_2, 2W_3) \).

3. Denote by \( \tau = \min\{t > 0 : W_t = a \text{ or } W_t = b\} \) the first time the Brownian motion process takes one of the values \( a \) or \( b \) \((a < 0 < b)\). Using the three martingales of the Brownian motion and the OST (lecture slide 182; don’t verify the conditions of the theorem),

   (a) find the distribution of \( W_\tau \);
   
   (b) compute the mean value \( \mathbb{E} \tau \);
   
   (c) compute the Laplace transform \( l_\tau(s) = \mathbb{E} e^{-s \tau}, s \geq 0 \), in the case when \( a = -1 \), \( b = 1 \);
   
   (d) use the result of part (c) to compute the mean \( \mathbb{E} \tau \) when \( a = -1 \), \( b = 1 \) (and compare the result with that for question (b)!).
   
   (e) Optional question: for enthusiasts only! Use the result of part (c) to compute the variance \( \text{Var}(\tau) \) when \( a = -1 \), \( b = 1 \).

4. Explain why the geometric Brownian motion \( X_t = e^{\mu t + \sigma W_t}, t \geq 0 \), is a Markov process. Derive the transition density

\[
p(s, x; t, y) = \frac{1}{dy} \mathbb{P}(X_t \in dy | X_s = x), \quad 0 < s < t, \quad x, y > 0,
\]

of the process \( \{X_t\} \) (lecture slides 224, 225 may be of help).
Homework Problems

1. Compute the joint densities of the random vectors:
   
   (a) \((2W_3, W_5)\);
   
   (b) \((W_2, 2W_3, W_5)\).

   *Hint:* (b) You may wish to use either the result of part (a) of this problem or that of Problem 2 from the tutorial.

2. Let \(\{X_t\}_{t \geq 0}\) be a continuous random process in continuous time, and \(\tau_x = \min\{t > 0 : X_t = x\}\) the first time the process \(\{X_t\}\) hits the level \(x > 0\). One can easily verify that, when \(X_t \equiv W_t\) is the standard Brownian motion process, then the density \(f_{\tau_x}\) of the distribution of \(\tau_x\) found in the theorem on lecture slide 219 has the form

   \[
   f_{\tau_x}(T) = \frac{x}{T} f_{X_T}(x), \quad T > 0. \tag{1}
   \]

   It turns out that the formula remains correct for some other processes as well. In particular, in the case of the arithmetic Brownian motion process \(X_t = \mu t + \sigma W_t\) the density of \(\tau_x\) is still given by (1).

   Using that observation, derive the density of \(\tau_2\) when \(\mu = 1\) and \(\sigma > 0\) is arbitrary. Sketch (on a common plot, on the time interval \([0, 3]\)) the graphs of the densities when (i) \(\sigma = 0.2\); (ii) \(\sigma = 0.5\); (iii) \(\sigma = 1\); (iv) \(\sigma = 2\).

   Comment on how the shape of the graph changes when \(\sigma\) increases and give an explanation (in your own words, in plain English) of the observed pattern of change.