Solutions to the *homework problems* are to be left in the 620-302 assignment box (**#181**) on the ground floor in the Richard Berry Building (north entrance). **Don't forget** to print your name, student ID, the subject name and code and your lecturer's name (K. Borovkov) on the first page of your solutions! All homework problems should be attempted. Only one (randomly chosen) of them will be marked. All material handed in must be on A4 size paper. Material on different sized paper will not be marked. The form and neatness of work can be considered in marking. Working and/or reasoning **must** be given to obtain full credit. The submission deadline is **5pm on Monday, 12 October 2009**.

Tutorial Problems

In what follows, $\{W_t\}_{t\geq 0}$ is a standard Brownian motion process.

1. Compute the mean function $m_X(t) = \mathbf{E}\, X_t$ and the covariance function $\gamma_X(s,t) = \mathrm{Cov}\,(X_s, X_t)$ when the process $\{X_t\}_{t\geq 0}$ is:

   (a) the arithmetic Brownian motion $X_t = \mu t + \sigma W_t$;

   (b) the geometric Brownian motion $X_t = e^{\mu t + \sigma W_t}$.

   In both cases, μ and σ are some real constants.

2. Compute the joint density of the random vector $(W_2, 2W_3)$.

3. Denote by $\tau = \min\{t > 0 : W_t = a \text{ or } W_t = b\}$ the first time the Brownian motion process takes one of the values a or b ($a < 0 < b$). Using the three martingales of the Brownian motion and the OST (lecture slide **182**; don't verify the conditions of the theorem),

   (a) find the distribution of W_τ;

   (b) compute the mean value $\mathbf{E}\,\tau$;

   (c) compute the Laplace transform $l_\tau(s) = \mathbf{E}\, e^{-s\tau}$, $s \geq 0$, in the case when $a = -1$, $b = 1$;

   (d) use the result of part (c) to compute the mean $\mathbf{E}\,\tau$ when $a = -1$, $b = 1$ (and compare the result with that for question (b)!).

   (e) *Optional question: for enthusiasts only!* Use the result of part (c) to compute the variance $\mathrm{Var}\,(\tau)$ when $a = -1$, $b = 1$.

4. Explain why the geometric Brownian motion $X_t = e^{\mu t + \sigma W_t}$, $t \geq 0$, is a Markov process. Derive the transition density

$$p(s, x; t, y) = \frac{1}{dy} \mathbf{P}\,(X_t \in dy |\, X_s = x), \qquad 0 < s < t, \quad x, y > 0,$$

   of the process $\{X_t\}$ (lecture slides **224, 225** may be of help).

Homework Problems

1. Compute the joint densities of the random vectors:

   (a) $(2W_3, W_5)$;

   (b) $(W_2, 2W_3, W_5)$.

   Hint: (b) You may wish to use either the result of part (a) of this problem or that of Problem 2 from the tutorial.

2. Let $\{X_t\}_{t\geq 0}$ be a continuous random process in continuous time, and $\tau_x = \min\{t > 0 : X_t = x\}$ the first time the process $\{X_t\}$ hits the level $x > 0$. One can easily verify that, when $X_t \equiv W_t$ is the standard Brownian motion process, then the density f_{τ_x} of the distribution of τ_x found in the theorem on lecture slide **219** has the form

$$f_{\tau_x}(T) = \frac{x}{T} f_{X_T}(x), \qquad T > 0. \tag{1}$$

   It turns out that the formula remains correct for some other processes as well. In particular, in the case of the arithmetic Brownian motion process $X_t = \mu t + \sigma W_t$ the density of τ_x is still given by (1).

   Using that observation, derive the density of τ_2 when $\mu = 1$ and $\sigma > 0$ is arbitrary. Sketch (on a common plot, on the time interval $[0, 3]$) the graphs of the densities when (i) $\sigma = 0.2$; (ii) $\sigma = 0.5$; (iii) $\sigma = 1$; (iv) $\sigma = 2$.

   Comment on how the shape of the graph changes when σ increases and give an explanation (in your own words, in plain English) of the observed pattern of change.