Solutions to the *homework problems* are to be left in the 620-302 assignment box (#**181**) on the ground floor in the Richard Berry Building (north entrance). **Don't forget** to print your name, student ID, the subject name and code and your lecturer's name (K. Borovkov) on the first page of your solutions! All homework problems should be attempted. Only one (randomly chosen) of them will be marked. All material handed in must be on A4 size paper. Material on different sized paper will not be marked. The form and neatness of work can be considered in marking. Working and/or reasoning **must** be given to obtain full credit. The submission deadline is **5pm on Monday, 19 October 2009**.

Tutorial Problems

In what follows, $\{W_t\}_{t\ge 0}$ is a standard Brownian motion process.

1. Let f_t and g_t be two simple processes on $[0,T]$: for a given (non-random) partition $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = T$,

$$\left.\begin{array}{l} f_t = X_k \\ g_t = Y_k \end{array}\right\} \quad \text{if} \quad t \in [t_{k-1}, t_k), \quad k = 1, 2, \ldots, n,$$

where X_k and Y_k are $\mathcal{F}_{t_{k-1}}$-measurable random variables, $\mathbf{E}\, X_k^2 < \infty$ and $\mathbf{E}\, Y_k^2 < \infty$. Verify that, for the Itô integrals $I_t(f) = \int_0^t f_s dW_s$ and $I_t(g) = \int_0^t g_s dW_s$,

$$\operatorname{cov}\left(I_t(f), I_t(g)\right) = \mathbf{E}\, I_t(f) I_t(g) = \int_0^t \mathbf{E}\left(f_s g_s\right) ds.$$

NB: Using the standard limiting procedure, one can extend the above result to the case of general adapted processes f_t and g_t satisfying the conditions $\int_0^T \mathbf{E}\, f_s^2 ds < \infty$ and $\int_0^T \mathbf{E}\, g_s^2 ds < \infty$. In fact, the above relation is a consequence of the "Itô isometry".

2. Let g_t be a non-random function on $[0,T]$ satisfying $\int_0^T g_t^2 dt < \infty$. Use Itô's formula to show that the process

$$Y_t = \exp\left\{\int_0^t g_s dW_s - \frac{1}{2}\int_0^t g_s^2 ds\right\}, \quad t \in [0,T],$$

is a martingale.

3. The price S_t of a risky asset evolves according to the stochastic differential equation (SDE)

$$dS_t = 0.2 S_t dt + S_t dW_t, \quad t \ge 0, \quad S_0 = 5.$$

(a) It is suspected that the SDE has a solution of the form $S_t = ce^{at+bW_t}$, where a, b and c are some constants. Use Itô's formula to verify this suspicion and find the values of the constants a, b and c in the solution.

(b) Show that the process $X_t = 1/S_t$ satisfies the SDE

$$dX_t = 0.8X_t dt - X_t dW_t, \quad X_0 = 0.2.$$

4. Compute the stochastic differential $d\cos(W_t)$.

Homework Problems

1. Put $X_t = t + W_t$, $t \geq 0$.

   (a) Apply Itô's formula to compute the stochastic differential de^{-2X_t}.

   (b) Is the process $Y_t = e^{-2X_t}$, $t \geq 0$, a martingale? Explain.

2. A "stochastic volatility" Heston model assumes that the "variance process" $\{V_t\}_{t\geq 0}$ follows the SDE

$$dV_t = (1 - V_t)dt + 2\sqrt{V_t}\, dW_t, \quad V_0 = 1.$$

   (a) Use Itô's formula to derive an SDE for the "volatility process"

$$Z_t = \sqrt{V_t}, \quad t \geq 0,$$

   and find the initial condition for the SDE (i.e. the value Z_0).

   (b) Use the product rule of Itô calculus to show that the process

$$Z_t = e^{-t/2}\left(1 + \int_0^t e^{s/2} dW_s\right)$$

   satisfies the SDE and the initial condition you derived in part (a).