

Solutions to the *homework problems* are to be left in the 620-302 assignment box (#181) on the ground floor in the Richard Berry Building (north entrance). **Don't forget** to print your name, student ID, the subject name and code and your lecturer's name (K. Borovkov) on the first page of your solutions! All homework problems should be attempted. Only one (randomly chosen) of them will be marked. All material handed in must be on A4 size paper. Material on different sized paper will not be marked. The form and neatness of work can be considered in marking. Working and/or reasoning **must** be given to obtain full credit. The submission deadline is **5pm on Monday, 26 October 2009**.

## Tutorial Problems

In what follows,  $\{W_t\}_{t \geq 0}$  is a standard Brownian motion process.

1. Let  $X_t := x_0 + t + 2W_t$  be a arithmetic Brownian motion process with a unit drift, starting at the point  $X_0 = x_0$ .
  - (a) Find the transition density  $p(s, x; t, y) := \frac{d}{dy} \mathbf{P}(X_t \leq y | X_s = x)$  of the process  $\{X_t\}$  ( $0 \leq s < t$ ).
  - (b) Write down the forward and backward Kolmogorov equations for the process  $\{X_t\}$ .
  - (c) Verify (by the direct computation of the respective derivatives) that the function  $u(t, y) := p(0, x_0; t, y)$ , where  $p(\dots)$  is the density you found in part (a), satisfies the forward equation you wrote down in part (b). What could you say about the initial condition that solution satisfies? [In other words: What happens to  $u(t, y)$  as  $t \rightarrow 0$ ?]
2. Let  $\{Y_t\}$  be an Ornstein-Uhlenbeck process satisfying

$$dY_t = -\alpha Y_t dt + \sigma dW_t, \quad \alpha, \sigma > 0, \quad Y_0 = x. \quad (1)$$

- (a) Write down the forward and backward Kolmogorov equations for the process  $\{Y_t\}$ .
- (b) Using the product rule of the Itô calculus, show that

$$Y_t := e^{-\alpha t} Z_{\sigma^2(e^{2\alpha t} - 1)/2\alpha}, \quad dZ_t = dW_t, \quad Z_0 = x,$$

is an Ornstein-Uhlenbeck process satisfying equation (1) [perhaps with a different Brownian motion process  $W_t^*$ ].

- (c) Use part (b) to derive the density of  $Y_t$ ,  $t > 0$ . Find the limit of the density as  $t \rightarrow \infty$  and comment on the significance of the existence of the limit from the viewpoint of the long-run behaviour of the Ornstein-Uhlenbeck process.

- (d) Using part (a), write down an ordinary differential equation for the stationary density  $\pi(y)$  of  $\{Y_t\}$  and solve it. Compare your answer with the limit you found in part (c). Comment on your findings.

*Hint.* (b) You may need to use the following “change of time rule”: for a differentiable increasing function  $f(t)$ ,  $t \geq 0$ , one has  $dW_{f(t)} = \sqrt{f'(t)} dW_t^*$ , where  $\{W_t^*\}$  is another Brownian motion. To see why the rule holds, observe that, for a small time increment  $\Delta t$ , one has  $f(t + \Delta t) \approx f(t) + f'(t)\Delta t$  and

$$W_{f(t)+f'(t)\Delta t} - W_{f(t)} \sim N(0, f'(t)\Delta t), \quad \sqrt{f'(t)} (W_{t+\Delta t} - W_t) \sim N(0, f'(t)\Delta t).$$

Or, more formally, note that  $W_{f(t)}$  has independent increments with

$$W_{f(t)} - W_{f(s)} \sim N(0, f(t) - f(s)), \quad 0 < s < t$$

(by the definition of the Brownian motion), and  $X_t = \int_0^t \sqrt{f'(u)} dW_u$  also has independent increments (why?), and as the integrand  $\sqrt{f'(u)}$  is non-random,

$$X_t - X_s = \int_s^t \sqrt{f'(u)} dW_u \sim N\left(0, \int_s^t \sqrt{f'(u)}^2 du\right) = N(0, f(t) - f(s))$$

since  $\int_s^t f'(u) du = f(t) - f(s)$ . So the two processes  $\{W_{f(t)}\}$  and  $\{X_t\}$  must have the same distribution.

## Homework Problem

- Let  $\{X_t : t \geq 0\}$  be a diffusion taking values in  $(0, \infty)$ , described by

$$dX_t = (1 - X_t) dt + \sqrt{2X_t} dW_t, \quad X_0 = x > 0.$$

- Write down the forward and backward Kolmogorov equations for the process.
- Use the method of differential equations to derive the stationary density of the process.
- Let  $0 < a < b < \infty$  and  $\tau = \min\{t \geq 0 : X_t = a \text{ or } X_t = b\}$  be the first time the process  $\{X_t\}_{t \geq 0}$  hits one of the barriers  $a, b$ . State an ordinary differential equation for the function

$$V(x) = \mathbf{P}(X_\tau = b | X_0 = x), \quad a < x < b.$$

Verify that the differential equation will be solved by  $V(x) = C_1 \text{Ei}(x) + C_2$ , where

$$\text{Ei}(x) = \gamma + \ln x + \sum_{n=1}^{\infty} \frac{x^n}{n \cdot n!}, \quad x > 0, \quad \gamma = 0.5772 \dots,$$

is the so-called *exponential integral*,  $C_i$  are constants. Use this result to find the function  $V(x)$  in this case. [*Hint:* The series representation for the function  $\text{Ei}(x)$  can be differentiated term-wise.]