Solutions to the *homework problem* **should NOT be left** in the 620-302 assignment box (#**181**) on the ground floor in the Richard Berry Building (north entrance). **DON'T** print your name, student ID, the subject name and code and your lecturer's name (K. Borovkov) on the first page of your solutions!

Tutorial Problems

In what follows, $\{W_t\}_{t\geq 0}$ is the standard Brownian motion process.

1. A diffusion process $\{X_t\}_{t\geq 0}$ is given by the following SDE:

$$dX_t = \frac{1}{2}X_t dt + dW_t.$$

   (a) Write down the backward and forward Kolmogorov equations for this process.

   (b) Use the backward Kolmogorov equation to derive an ordinary differential equation for the probability

$$V(x) := \mathbf{P}\left(X_\tau = b \mid X_0 = x\right),$$

   where

$$\tau := \min\{t \geq 0 : \ X_t = a \text{ or } X_t = b\}, \quad a < x < b,$$

   is the first time the process hits one of the barriers a and b. Specify appropriate boundary conditions for the equation and solve it.

   (c) Sketch the plots of $V(x)$, $x \in [a, b]$, when (i) $(a, b) = (-2, 2)$; (ii) $(a, b) = (-1, 1)$; (iii) $(a, b) = (0, 2)$; (iv) $(a, b) = (2, 4)$. Comment briefly on the differences in shape.

   (d) Find $\mathbf{P}\left(\max_{t\geq 0} X_t > b \mid X_0 = x\right)$, $\quad x, b \in \mathbf{R}$.

   Hint. (b)–(d) You may find it convenient to use the standard normal distribution function in presenting your results.

2. A diffusion process $\{X_t\}_{t\geq 0}$ given by the SDE

$$dX_t = (-\gamma_1 X_t + \gamma_2(1 - X_t))dt + \sqrt{X_t(1 - X_t)}\, dW_t, \qquad X_t \in (0, 1),$$

   with $\gamma_j \geq 0$ arises in the Wright-Fisher model for the fluctuation of gene frequency without selection differences. One can show that the process has a stationary distribution with a density $\pi(y)$.

   (a) Write down the forward and backward Kolmogorov equations for the process $\{X_t\}$.

(b) Consider the special case of equal mutation rates $\gamma_1 = \gamma_2 = 1$. Write down an ordinary differential equation for the stationary density and solve the equation. Plot the stationary density $\pi(y)$.

Homework Problem

1. A diffusion process $\{X_t\}_{t\geq 0}$ given by the SDE

$$dX_t = X_t(1 - X_t)dt + \sqrt{X_t(1 - X_t)}\, dW_t, \qquad X_t \in (0, 1),$$

corresponds to the Wright-Fisher gene frequency model involving selection only.

(a) Write down the backward and forward Kolmogorov equations for the process $\{X_t\}$.

(b) Assuming that the initial frequency was $X_0 = x \in (0, 1)$, derive the probability of the eventual fixation of the gene frequency at one (meaning that $X_t = 1$ for all $t \geq \tau$ for some time τ). That event occurs if the process X_t hits the boundary point 1 before hitting point 0.