Tutorial 1 (solutions)

Solutions to tutorial problems.

1. If $V > X^* + Y^*$ (V denotes the time $t=0$ price of the claim $X+Y$),

- Sell the claim $X+Y$ at price V;
- form replicating portfolios for X (costs X^*) and Y (costs Y^*);
- spend the remaining money $(V - (X^* + Y^*) > 0)$ on bonds;
- then at the end of the day, the portfolios will cover the claim (your obligations! – covering each of the components), and you'll keep the bond. Arbitrage.

If $V < X^* + Y^*$,

- Form portfolios replicating $-X$ and $-Y$ (explain what this means!), this will give you $X^* + Y^*$;
- Buy the claim $X+Y$ at the price V;
- Use the residual $(X^* + Y^* - V > 0)$ to buy bonds;
- at the end of the day, you'll get the claim values (X & Y) which would compensate the portfolios' values, and you'll keep the bonds. Arb'ge

2. From slide (27):

$$C(K) = \frac{1}{1+r}\left[p^*(uS_0 - K)^+ + (1-p^*)(dS_0 - K)^+\right],$$

$$0 < d < u, \quad p^* \in (0,1).$$

Note:

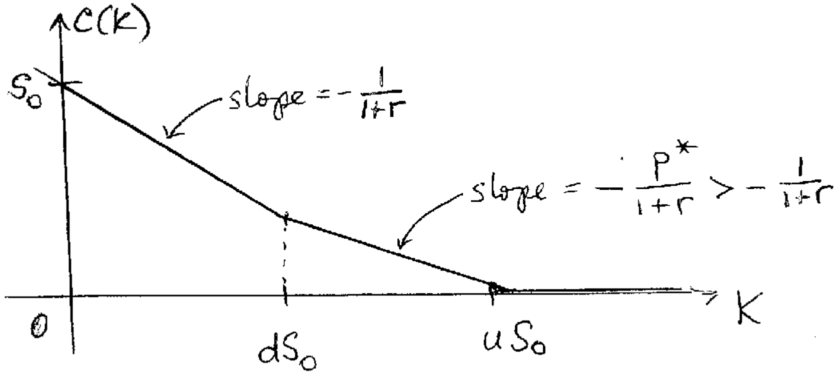
- if $K \geq uS_0$, both $(\cdot)^+ = 0$, so $C(K) = 0$;
- if $dS_0 \leq K \leq uS_0$, of the two $(\cdot)^+$'s, only the first one is present, and it's equal to $uS_0 - K$ (linear f'n), so $C(K) = \frac{p^*}{1+r}(uS_0 - K)$ in this range;
- if $K \leq dS_0$, both $(\cdot)^+$ are present, and

$$C(K) = \frac{1}{1+r}\left[p^*(uS_0 - K) + (1-p^*)(dS_0 - K)\right]$$

$$= \frac{1}{1+r} \times \left(\frac{(1+r-d)u}{u-d} + \frac{(u-(1+r))d}{u-d}\right) S_0 - \frac{K}{1+r}$$

$$= S_0 - \frac{K}{1+r}.$$

Thus:



Common features:

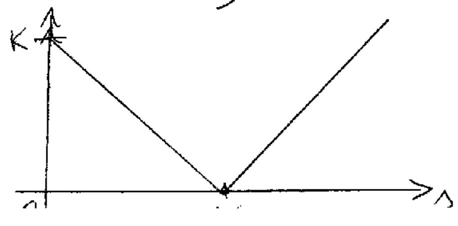
(i) $C(0) = S_0$ (call = can buy the stock at price $K=0$, i.e. just receive it!)

(ii) $C(K)$ is a decreasing function (the bigger K the smaller the payoff, so should cost less!);

(iii) $C(K) \to 0$ as $K \to \infty$ (as the payoff vanishes).

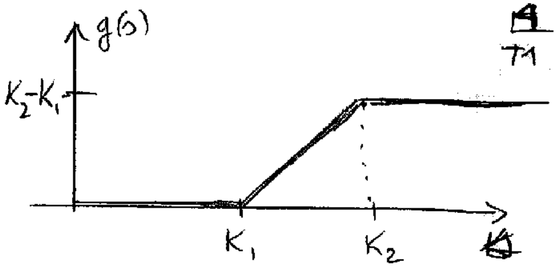
<u>3.</u> $X = C + P$, so

$g(s) = (s-K)^+ + (s-K)^-$



<u>4.</u> (a) $X = C_1 - C_2$, so the payoff is

$g(s) = (s-K_1)^+ - (s-K_2)^+$



(b) As $X = C_1 - C_2 < C_1$, the spread must be cheaper (the price is given by $\frac{1}{1+r}E^*(X) < \frac{1}{1+r}E^*(C_1)$).

(c) As $S_0 = 5$, $dS_0 = 4$, $uS_0 = 6$, we get $d = \frac{4}{5} = 0.8$, $u = \frac{6}{5} = 1.2$ (note: $d < 1+r = 1.1 < u$, so NA & can use the pricing formula),

$$p^* = \frac{1+r-d}{u-d} = \frac{1.1-0.8}{1.2-0.8} = \frac{0.3}{0.4} = \frac{3}{4},\ 1-p^* = \frac{1}{4},$$

and hence the price

$$X^* = \frac{1}{1+r}E^*(X) = \frac{1}{1+r}\left[p^* g(uS_0) + (1-p)^* g(dS_0)\right]$$

$g(uS_0) = g(6) = K_2 - K_1 = 2$ (as $6 > K_2$); $g(dS_0) = g(4) = 4 - K_1 = 1$ (as $K_1 < 4 < K_2$)

$$= \frac{1}{1.1}\left[\frac{3}{4}\times 2 + \frac{1}{4}\times 1\right] = \frac{7}{4.4} = \frac{35}{22} \approx 1.59$$

(d) $$\Delta = \frac{g(6)-g(4)}{6-4} = \frac{2-1}{2} = \frac{1}{2},$$

$$b = \frac{1.2\times g(4) - 0.8\times g(6)}{1.1\times 0.4} = -\frac{10}{11}.$$

Verification:

$V_1(u) = \frac{1}{2}\times 6 - \frac{10}{11}\times 1.1 = 3-1=2$, ok. g(6)

$(\Delta \times u S_0 - b\times(1+r))$ — same values as

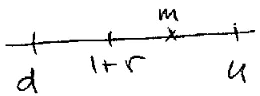
$V_1(d = \frac{1}{2}\times 4 - \frac{10}{11}\times 1.1 = 2-1=1$, ok. g(4)

5. (a) NA $\iff \exists\, p_j^* > 0$, $\sum_{j=1}^{3} p_j^* = 1$ s.t.

$$S_0 = \frac{1}{1+r}\left(p_1^* d S_0 + p_2^* m S_0 + p_3^* u S_0\right)$$

or, eq'ly,

$$p_1^* d + p_2^* m + p_3^* u = 1+r$$



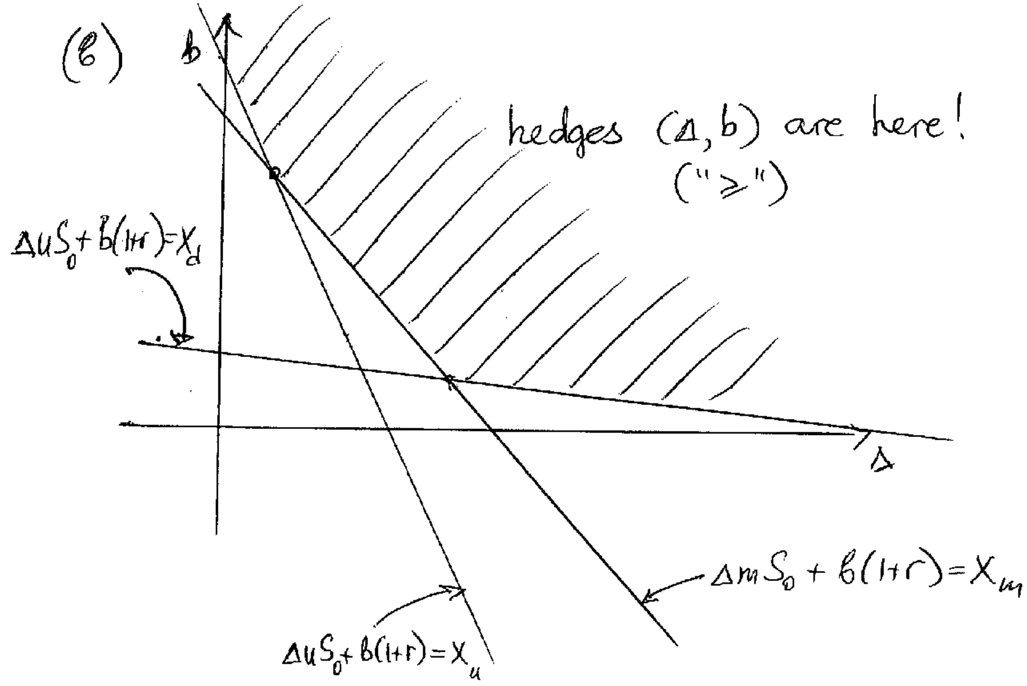
But note: dividing by $p_1^* + p_3^* = 1 - p_2^*$, this is

$$\underbrace{\frac{p_1^*}{p_1^*+p_3^*}}_{p} d + \underbrace{\frac{p_3^*}{p_1^*+p_3^*}}_{1-p} u = \underbrace{\frac{1+r}{1-p_2^*} - \frac{p_2^*}{1-p_2^*} m}_{\text{still } \in (d,u) \text{ for } p_2^*>0 \text{ small enough!}}$$

So, as in lectures, can always find a $p \in (0,1)$ s.t.

$pd + (1-p)u = RHS \longrightarrow p_j^*$!!

(b)



Replication $\iff \exists\, (\Delta, b) \in$ all three lines, i.e. $\in$ their intersection $= \emptyset$, nope!