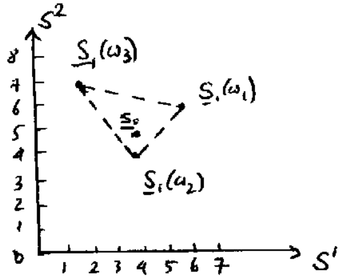
620302/104

Tutorial 2 (solutions)

1. (a)



(b) NA $\Leftrightarrow$ $\exists$ EMM $P^* \Leftrightarrow \underline{S}_0 \in$ rel. interior of the conv$(\underline{S}_1(\omega_1), \underline{S}_1(\omega_2), \underline{S}_1(\omega_3))$, which is the case (see pic. in (a)).

(c) Need to find prob's p_1^*, p_2^*, p_3^* s.t.

$$\underline{S}_0 = E^*\left(\frac{\underline{S}_1}{1+r}\right) = p_1^* \underline{S}_1(\omega_1) + p_2^* \underline{S}_1(\omega_2) + p_3^* \underline{S}_1(\omega_3)$$

or, component-wise, (using p_j inst'd of p_j^*)

$$\begin{cases} 4 = 6p_1 + 4p_2 + 2p_3 \\ 5 = 6p_1 + 4p_2 + 7p_3 \\ 1 = p_1 + p_2 + p_3 \end{cases}$$
$\leftarrow$ this is a prob'ty distr'n!

$\to 1 = 5p_3$ (from eq's 1+2), $p_3 = 0.2$,

$\to p_1 + p_2 = 0.8$ (from eq'n 3), $p_2 = 0.8 - p_1$

$\to 4 = 6p_1 + 4(0.8 - p_1) + 2 \times 0.2$ (from eq'n 1)

so $p_1^* = 0.2$, $p_2^* = 0.6$, $p_3^* = 0.2$.

Since this is obviously a unique sol'n, the prob'ty P^* is unique $\to$ the market is complete.

(d) For $X = (S_1^1 - K)^+ = (S_1^1 - 5)^+$,

$$X^* = E^*\left(\frac{X}{1+r}\right) = p_1^*(S_1^1(\omega_1) - 5)^+ + p_2^*(S_1^1(\omega_2) - 5)^+ + p_3^*(S_1^1(\omega_3) - 5)^+$$

$$= 0.2 \times \underbrace{(6-5)^+}_{=1} + 0.6 \times \underbrace{(4-5)^+}_{=0} + 0.2 \times \underbrace{(2-5)^+}_{=0} = 0.2.$$

(e) Find (Δ^1, Δ^2, b) s.t. $\forall \omega \in \Omega$, $\Delta^1 S_1^1(\omega) + \Delta^2 S_1^2(\omega) + b(1+r) = X(\omega)$, i.e.

$$\begin{array}{ll} \omega_1: & 6\Delta^1 + 6\Delta^2 + b = 1 \\ \omega_2: & 4\Delta^1 + 4\Delta^2 + b = 0 \\ \omega_3: & 2\Delta^1 + 7\Delta^2 + b = 0 \end{array}$$

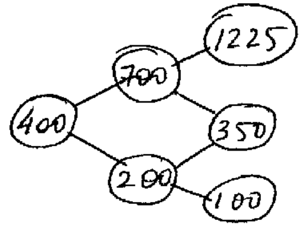
$\to \frac{1}{2}b = -1$ (eq's 1+2), $b = -2$;

$\to \begin{cases} 4\Delta^1 + 4\Delta^2 = 2 \\ 2\Delta^1 + 7\Delta^2 = 2 \end{cases} \to 10\Delta^2 = 2$, $\Delta^2 = 0.2$; $\Delta^1 = (2 - 4\Delta^2)/4 = 0.3$,

$(\Delta^1, \Delta^2, b) = (0.3, 0.2, -2)$.

Yes, we do need stock 2 (as this portfolio is a unique solution to the linear system).

2. (a) Stock Prices:



$d = 0.5 < 1+r = 1.25 < u = 1.75$, OK.

$$p^* = \frac{1+r-d}{u-d} = \frac{1.25 - 0.5}{1.75 - 0.5} = 0.6,$$

$1 - p^* = 0.4$.

The put's payoff: $(S_2 - 450)^-$, so

$$\begin{tikzcd}[column sep=small]
 & & \boxed{(1225-450)^- = 0} \\
 & \boxed{\frac{0\times 0.6+0.4\times 100}{1.25}=32} \arrow[ur, dash] \arrow[dr, dash] & \\
\boxed{\frac{0.6\times 32+0.4\times 160}{1.25}=66.56} \arrow[ur, dash] \arrow[dr, dash] & & \boxed{(350-450)^- = +100} \\
 & \boxed{\frac{0.6\times 100+0.4\times 350}{1.25}=160} \arrow[ur, dash] \arrow[dr, dash] & \\
 & & \boxed{(100-450)^- = +350}
\end{tikzcd}$$

The put price $P = 66.56$.

(b) • At $t=1$, $S_1 = 700$: $(\Delta_2, b_2) = \left(-\frac{4}{35}, \frac{448}{5}\right)$.

$$\Delta_2 = \frac{0-100}{1225-350} = -\frac{100}{875} = -\frac{4}{35} \quad (\approx -0.114)$$

$$b_2 = \frac{1.75\times 100 - 0.5\times 0}{1.25^2 \times 1.25} = \frac{448}{5} = 89.6$$

• At $t=1$, $S_1 = 200$: $(\Delta_2, b_2) = (-1, 288)$.

$$\Delta_2 = \frac{100-350}{350-100} = -1$$

$$b_2 = \frac{1.75\times 350 - 0.5\times 100}{1.25^2\times 1.25} = 288$$

• At $t=0$: $(\Delta_1, b_1) = \left(-\frac{32}{125}, 168.96\right)$

$$\Delta_1 = \frac{32-160}{700-200} = -\frac{32}{125} = -0.256$$

$$b_1 = \frac{1.75\times 160 - 0.5\times 32}{1.25\times 1.25} = 168.96$$

(c) We just have to verify that at time $t=1$.

$$\Delta_1 S_1 + b_1(1+r) = \Delta_2 S_1 + b_2(1+r)$$

• If $S_1 = 700$, $\Delta_1 S_1 + b_1(1+r) = -0.256\times 700 + 168.96\times 1.25 = 32$
$\Delta_2 S_1 + b_2(1+r) = -\frac{4}{35}\times 700 + 89.6\times 1.25 = 32$

• If $S_1 = 200$, $\Delta_1 S_1 + b_1(1+r) = -0.256\times 200 + 168.96\times 1.25 = 160$
$\Delta_2 S_1 + b_2(1+r) = -1\times 200 + 288\times 1.25 = 160$

3 (a) As $C = S_0 N(h) - Ke^{-rT}N(h-\sigma\sqrt{T})$, $h = \frac{\ln(S_0/K)+(r+\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$,

$$\Delta = \frac{\partial C}{\partial S_0} = N(h) + S_0\frac{\partial h}{\partial S_0}N'(h) - Ke^{-rT}\frac{\partial h}{\partial S_0}N'(h-\sigma\sqrt{T})$$

$$\boxed{\text{using } \frac{\partial h}{\partial S_0} = \frac{1}{\sigma\sqrt{T}S_0}, \quad N'(h) = \frac{1}{\sqrt{2\pi}}e^{-h^2/2}}$$

$$= N(h) + \frac{1}{\sigma\sqrt{2\pi T}}\left[e^{-h^2/2} - \frac{K}{S_0e^{rT}}e^{-(h-\sigma\sqrt{T})^2/2}\right]$$

$$= N(h) + \frac{e^{-h^2/2}}{\sigma\sqrt{2\pi T}}\left[1 - \frac{K}{S_0e^{rT}}\underbrace{e^{h\sigma\sqrt{T}}}_{=\exp\{\ln(S_0/K)+rT+\frac{\sigma^2T}{2}\}}e^{-\sigma^2T/2}\right]$$

$$= N(h).$$

(b) $$\Gamma = \frac{\partial^2 C}{\partial S_0^2} = \frac{\partial}{\partial S_0}N(h) = N'(h)\frac{\partial h}{\partial S_0} = \frac{e^{-h^2/2}}{\sqrt{2\pi T}\,\sigma S_0}.$$

(c) $$\mathcal{V} = \frac{\partial C}{\partial \sigma} = S_0N'(h)\frac{\partial h}{\partial \sigma} - Ke^{-rT}N'(h-\sigma\sqrt{T})\left(\frac{\partial h}{\partial \sigma} - \sqrt{T}\right)$$

as in (a)

$$= \frac{S_0e^{-h^2/2}}{\sigma\sqrt{2\pi T}}\frac{\partial h}{\partial \sigma}\underbrace{\left[1 - \frac{K}{S_0e^{rT}}e^{h\sigma\sqrt{T}}e^{-\sigma^2T/2}\right]}_{=0 \text{ as in (a)}}$$

$$+ \sqrt{T}Ke^{-rT}\frac{e^{-h^2/2}}{\sqrt{2\pi}}\exp\left\{\ln(S_0/K) + rT + \frac{\sigma^2T}{2} - \frac{\sigma^2T}{2}\right\}$$

$$= S_0\sqrt{T}\frac{e^{-h^2/2}}{\sqrt{2\pi}}.$$

(d) From PCP: $S_0 + P - C = Ke^{-rT}$ (Slide 36) $_{(t=0)}$, $\frac{15}{T2}$

$$\frac{\partial P}{\partial S_0} = \frac{\partial C}{\partial S_0} - \frac{\partial S_0}{\partial S_0} = N(h) - 1 < 0 \quad \text{for all } h,$$

whereas $\frac{\partial C}{\partial S_0} = N(h) > 0$ for all h.