# Tutorial 3

Solutions to tutorial problems.

**1** (a) $\{1\} = \bar{A}_1 \in \mathcal{F}$ by A2 (as $A_1 \in \mathcal{F}$)

(b) $[1/4, 1/2) = A_2 \setminus A_4$ ($\equiv A_2 \cap \bar{A}_4$) $\in \mathcal{F}$

(c) $\{0\} = \bigcap_{n=k}^{\infty} A_n$ (for any $k \geq 1$),

so $\in \mathcal{F}$ (by A2, A3: $\bigcap A_n = \overline{(\bigcup \bar{A}_n)}$)

(d) Perhaps, but can't tell from the conditions (if $\mathcal{F}$ were generated by $\{A_n\}_{n\geq 1}$, then the answer would be NO; but our $\mathcal{F}$ may be bigger, we only know that all $A_n \in \mathcal{F}$).

**2** First recall: for any $x \in \mathbb{R}$, the singleton $\{x\} \in \mathcal{B}$ (i.e. is a Borel set), slide 91.

By definition, for a RV $X$ we must have $\{X \in B\} \in \mathcal{F}$ for any Borel set $B$; so, in particular, $\{X \in \{x\}\} = \{X = x\}$ must be an event, i.e. $\in \mathcal{F}$.



(a) Thus in this case either $\{X = x\} = \emptyset$ (impossible!) or $\{X = x\} = \Omega$ (certain event!). This can only mean that $X \equiv x_0$ for some const. value $x_0$. That is, in case $\mathcal{F} = \{\emptyset, \Omega\}$ all RV are just constants (non-random).

(b) In this case, $\{X \in B\} \in \mathcal{F}$ always (as $\mathcal{F}$ contains all subsets of $\Omega$!). So any function $X(\omega)$ on $\Omega$ will satisfy the condition and can be called a random variable.

(c) Now, for any $x \in \mathbb{R}$,

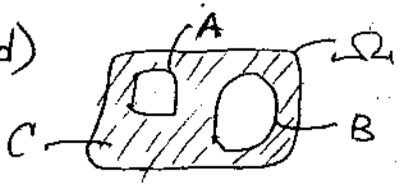
$$\{X = x\} = \begin{cases} \text{either } \emptyset \\ \text{or } A \\ \text{or } \bar{A} \\ \text{or } \Omega \end{cases}$$

- If $\{X = x\} = \Omega$ holds, it means $X = x$ always, i.e. $X$ is a constant (not random!).
- If $\{X = x\} = A$ holds, it means that $X(\omega) = x$ for all $\omega \in A$, i.e. is constant

("flat") on this set. Outside $A$ (i.e. in $\bar{A}$) it assumes different values. In fact, there will be just one different value for $X$: take any $\omega' \in \bar{A}$ and let $x' = X(\omega')$. Can't have $\{X = x'\} = \emptyset$ (as $\omega'$ is there!), can't have $\{X = x'\} = A$ (as $X = x$ on $A$), can't have $\{X = x'\} = \Omega$ (— " —). So must have $\{X = x'\} = \bar{A}$.

Therefore any RV on such an $\mathcal{F}$ will be "flat" on $A$ and "flat" on $\bar{A}$ (i.e. assuming at most two different values).

(d)




$\mathcal{F} = \{\emptyset, A, B, C, A \cup B\ (= \bar{C}), A \cup C\ (= \bar{B}), B \cup C\ (= \bar{A}), \Omega\}$

Here: any RV $X$ will be:

Constant (flat) on $A$
— " — $B$
— " — $C$

(i.e. can assume at most three diff't values). Similar to part (c) argument.



3 (a) Since $A = \underbrace{(AB)}\cup\underbrace{(A\bar{B})}$, we get ← disjoint events!

independence

$$P(A) = P(AB) + P(A\bar{B}) \overset{\text{independence}}{=} P(A)P(B) + P(A\bar{B}), \text{ so}$$

$$P(A\bar{B}) = P(A) - P(A)P(B) = P(A)(1 - P(B)) = P(A)P(\bar{B}),$$

so – yes!

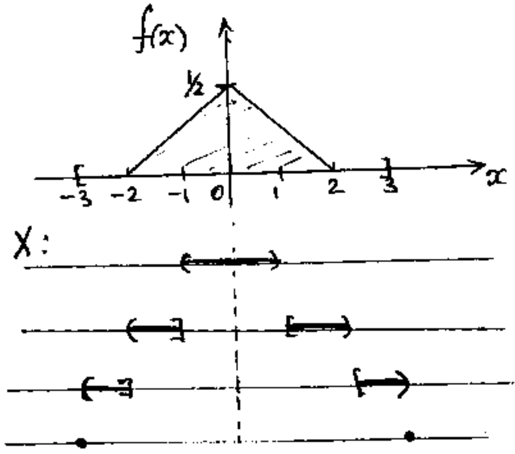
(b) Yes – by (a) (apply it to $A_1 = \bar{B}$, $B_1 = A$)

(c) No. If they were we would have:

$$0 = P(\emptyset) = P(A\bar{A}) \overset{!}{=} P(A)P(\bar{A}) = P(A)(1 - P(A)),$$

i.e. either $P(A) = 0$ or $P(A) = 1$ – but in both cases $A$ is in fact not random anymore.

4.




Note:

$\{X = 0\} = (-1, 1) =: A_0$

$\{X = 1\} = (-2, -1] \cup [1, 2) =: A_1$

$\{X = 2\} = (-3, -2] \cup [2, 3) =: A_2$

$\{X = 3\} = \{-3\} \cup \{3\} =: A_3$

(a) Obviously all $A_j \in \mathcal{F}$, $j = 0, 1, 2, 3$.

By A3, $\mathcal{F}$ will also contain all possible unions of the $A_j$'s ( $A_0 \cup A_1$, $A_0 \cup A_2$, $A_0 \cup A_3$, $A_1 \cup A_2$, $A_1 \cup A_3$, $A_2 \cup A_3$,

$A_0 \cup A_1 \cup A_2$, $A_1 \cup A_2 \cup A_3$, $A_0 \cup A_1 \cup A_3$, $A_0 \cup A_2 \cup A_3$, $\Omega$) – and note that their complements have already been listed! Any set of the form $\{X \in B\}$ will be one of these unions. So this completes the discription of $\mathcal{F} = \sigma(X)$: all possible unions of the $A_j$'s (incl. the "empty union" and the $A_j$'s themselves).

(b) Possible values are: 0, 1, 2, 3.

Clearly, $P(X=3) = P(X=2) = 0$ (as $P(X<2) = P(\omega \in (-2,2)) = 1$).

And

$$P(X=0) = P(\omega \in (-1,1)) = \int_{-1}^{1} f(x)dx \overset{\text{by symmetry}}{=} 2\int_0^1 f(x)dx$$

$$= 2\int_0^1 \frac{2-x}{4}dx = \int_0^1 \left(1 - \frac{x}{2}\right)dx = \left[x - \frac{x^2}{4}\right]_0^1 = \frac{3}{4},$$

$$P(X=1) = 1 - P(X=0) = \frac{1}{4}$$

or just find the areas under the "density curve" computing the areas of the resp. triangles...