Tutorial 4 (solutions)

1. $\operatorname{cov}(\tilde X, X-\tilde X) = E\,\tilde X(X-\tilde X) - E\tilde X\cdot E(X-\tilde X)$

$$\overset{\boxed{CE\text{-}1},\boxed{CE4}}{=} E\,E\left[\tilde X(X-\tilde X)\,|\,Y\right] - E\tilde X\cdot\left(EX - \underbrace{E\tilde X}_{=E\,E(X|Y)\overset{\boxed{CE4}}{=}EX}\right)$$

$$\overset{\boxed{CE\text{-}2}}{=} E\left[\tilde X\,\underbrace{E\left(X-\tilde X\,|\,Y\right)}_{\overset{\boxed{CE\text{-}1}}{=}}\right] - E\tilde X\cdot\left(\underbrace{EX-EX}_{=0}\right) = 0$$

$$\underbrace{E(X|Y)}_{=\tilde X} - E(\tilde X|Y) \overset{\boxed{CE\text{-}2}}{=} \tilde X - \tilde X = 0$$

2. (a) $E(N_{t+s}\,|\,\mathcal F_t) = E(N_t + N_{t+s} - N_t\,|\,\mathcal F_t)$

$$\overset{\boxed{CE\text{-}1}}{=} \underbrace{E(N_t\,|\,\mathcal F_t)}_{\overset{\boxed{CE\text{-}2}}{=}N_t} + \underbrace{E(N_{t+s}-N_t\,|\,\mathcal F_t)}_{=E(N_{t+s}-N_t)}$$

since N_t is $\mathcal F_t$-measurable

$\boxed{CE3}$ since $\{N_t\}$ has indep't increments: $N_{t+s}-N_t$ is indep't of the history on $[0,t]$;

$= EN_s = \lambda s$ (as N_s is Poisson with par'r λs)

$= N_t + \lambda s$

(b) $E(N_{t+s}^2\,|\,\mathcal F_t) = E\left((N_t + N_{t+s} - N_t)^2\,|\,\mathcal F_t\right)$

$$\overset{\boxed{CE1}}{=} \underbrace{E(N_t^2\,|\,\mathcal F_t)}_{\overset{\boxed{CE2}}{=}N_t^2,\text{ as in (a)}} + 2\underbrace{E\left(N_t(N_{t+s}-N_t)\,|\,\mathcal F_t\right)}_{\overset{\boxed{CE2}}{=}N_t\underbrace{E(N_{t+s}-N_t\,|\,\mathcal F_t)}_{=\lambda s\text{ as in (a)}}} + \underbrace{E\left((N_{t+s}-N_t)^2\,|\,\mathcal F_t\right)}_{\overset{\boxed{CE3}}{=}E(N_{t+s}-N_t)^2}$$

$$= N_t^2 + 2\lambda s N_t + E N_s^2 = N_t^2 + 2\lambda s N_t + (\lambda s)^2 + (\lambda s)$$

as $N_s \overset{d}{=} N_{t+s} - N_t$, same distr^n

$$= (N_t + \lambda s)^2 + \lambda s$$

(c) $E(N_s \mid \mathcal{F}_t) \overset{\boxed{CE\,2}}{=} N_s$
$E(N_s^2 \mid \mathcal{F}_t) = N_s^2$ } as in both cases, the RV is $\mathcal{F}_t$-measurable (if we know the "history" of the process on $[0,t]$ and $s \le t$, then we know the value of N_s - and hence of N_s^2 as well).

(d) $E(N_s \mid N_t) = \eta(N_t)$, where

$\eta(n) = E(N_s \mid N_t = n) = ns/t$ as the cond'l distr^n of N_s given $N_t = n$ is $Bi(n, s/t)$ $(s \le t)$, done in 620-301 (easy: for $k = 0, 1, \dots, n$,

$$P(N_s = k \mid N_t = n) = \frac{P(N_s = k, N_t = n)}{P(N_t = n)} = \frac{P(N_s = k, N_t - N_s = n-k)}{P(N_t = n)}$$

$$\overset{\text{indep}^t \text{ increments}}{=} \frac{P(N_s = k)\,P(N_t - N_s = n-k)}{P(N_t = n)} = \frac{e^{-\lambda s}\frac{(\lambda s)^k}{k!} \times e^{-\lambda(t-s)}\frac{(\lambda(t-s))^{n-k}}{(n-k)!}}{e^{-\lambda t}\frac{(\lambda t)^n}{n!}}$$

$$= \binom{n}{k}\left(\frac{s}{t}\right)^k \left(1 - \frac{s}{t}\right)^{n-k}\Big),$$ so $E(N_s \mid N_t) = \frac{s}{t} N_t$.

Similarly, $E(N_s^2 | N_t) = \eta(N_t)$, where
$\eta(n) = E(N_s^2 | N_t = n) = n\frac{s(t-s)}{t^2} + n^2\frac{s^2}{t^2}$ for the
same reason, so $E(N_s^2 | N_t) = \frac{s(t-s)}{t^2} N_t + \frac{s^2}{t^2} N_t^2$.

(e) $\varphi(u) = \sum_{k=0}^{\infty} e^{iuk} P(N_t = k) = \sum_{k=0}^{\infty} e^{iuk} e^{-\lambda t} \frac{(\lambda t)^k}{k!}$

$= e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(\lambda t e^{iu})^k}{k!} = \underline{\exp\{\lambda t (e^{iu} - 1)\}}$.

<u>3</u> By independence, for the ch.f's:
$\varphi_Z(t) = \varphi_X(t)\varphi_Y(t)$ (slide 134), so, using
the result of <u>2(e)</u>,

$e^{3.5(e^{it}-1)} = e^{2(e^{it}-1)} \varphi_Y(t)$, or

$\varphi_Y(t) = e^{1.5(e^{it}-1)}$. This is the ch.f. of
the Poisson (1.5) distribution, and since
ch.f's specify distrns uniquely, we conclude
that Y is Poisson with rate $\lambda_Y = 1.5$.

4 (a) $E(S_N) \overset{\boxed{CE4}}{=} E\,\underbrace{E(S_N|N)}_{=\eta(N)}$, where

$$\eta(n) = E(S_N|N=n) = E(S_n|N=n) \underset{\text{as } S_n \text{ and } N \text{ are indep't}}{=} ES_n = nEX_1$$

$$\underset{\text{slide } 137}{=} n\,\frac{h'(0)}{i} = -inh'(0), \text{ so } \quad ES_N = E\eta(N) = -ih'(0)EN \underset{\text{sl. } 130}{=} -ih'(0)g'(1)$$

$$ES_N^2 \overset{\boxed{CE4}}{=} E\,\underbrace{E(S_N^2|N)}_{=\eta(N)}, \text{ where}$$

$$\eta(n) = E(S_N^2|N=n) \underset{\text{as above}}{=} ES_n^2 = var(S_n) + (ES_n)^2$$

$$= n\,var(X_1) + (nEX_1)^2 = n\left(EX_1^2 - (EX_1)^2\right) + n^2(EX_1)^2$$

$$\underset{\text{sl. } 137}{=} n\left(-h''(0) - (-ih'(0))^2\right) + n^2(-ih'(0))^2$$

$$= n\left(h'(0)^2 - h''(0)\right) - n^2 h'(0)^2, \qquad (\text{as } (-i)^2 = i^2 = -1)$$

So $ES_N^2 = E\eta(N) = (h'(0)^2 - h''(0))EN - h'(0)^2 EN^2$

$$\underset{\text{sl. } 130}{\overset{*}{=}} (h'(0)^2 - h''(0))g'(1) - h'(0)^2\left(g''(1) + g'(1)\right)$$

$$= -h''(0)g'(1) - h'(0)^2 g''(1).$$

$EN^2 \overset{*}{=} EN(N-1) + EN$

(b) $\varphi(t) = E\left[E(e^{itS_N}|N)\right] \overset{*}{=} E\,h(t)^N = g(h(t)).$

* as above, $E(e^{itS_N}|N=n) = Ee^{itS_n} \underset{\text{by indep'ce, } 134}{=} h(t)^n$

(c) Since $g(s) = Es^N = \sum_{k=0}^{\infty} s^k p q^k = p\sum_{k=0}^{\infty}(sq)^k = \frac{p}{1-sq}$,

$h(t) = Ee^{itX_1} = re^{it} + (1-r)$, $\varphi(t) = g(h(t)) = p/(1 - qre^{it} - q(1-r))$

$$= \frac{p}{p + qr - qre^{it}} = \frac{p'}{1 - q'e^{it}}, \quad p' = \frac{p}{p+qr} \in (0,1),\ q' = 1 - p';$$ geometric with p' (cf. $g(s)$)