Tutorial 5 (solutions)

1. (a) $\varphi_X(t) = Ee^{itX} = \frac{1}{\Gamma(r)} \int_0^\infty e^{itx} e^{-x} x^{r-1} dx$

$$= \frac{1}{(1-it)^r \Gamma(r)} \underbrace{\int_0^\infty e^{-(1-it)x} [(1-it)x]^{r-1} d[(1-it)x]}_{=\Gamma(r)}$$

$= \Gamma(r)$ (changing var's $x \to (1-it)x = y$, as if $(1-it)$ is real; cf. slide 140)

$$= (1-it)^{-r}$$

(b) $\varphi_X^{(k)}(t) = \frac{d^{k-1}}{dt^{k-1}} \varphi_X'(t) = \frac{d^{k-1}}{dt^{k-1}} (-r)(-i)(1-it)^{-r-1}$

$$= ir \frac{d^{k-1}}{dt^{k-1}} (1-it)^{-r-1} = \ldots = i^k r(r+1)\cdots(r+k-1)$$

$= i^k EX^k$, hence $EX^k = r(r+1)\cdots(r+k-1)$, $k = 1, 2, \ldots$

(7), 137

(c) Not stable: e.g. $\varphi_X(t)\varphi_X(t) = (1-it)^{-2r} \neq$ $\neq \varphi_X(at)e^{itb} = (1-iat)^{-r} e^{itb}$ whatever the choice of a, b is (cf. bottom of 158). Alternatively, $\varphi_X(t)$ is not of the form $e^{it\mu - c|t|^\alpha(1-i\ldots}$

But infinitely divisible as for any $n = 1, 2, \ldots$ $(\varphi_X(t))^{1/n} = (1-it)^{-r/n}$ is a ch.f. again (that of $\gamma(r/n, 1)$).

(d) $\varphi_{-X}(t) \underset{(4),(136)}{=} \varphi_X(-t) = (1+it)^{-r}$, so that

$$\varphi_Y(t) \underset{(5),(136)\text{ as } X_1, -X_2 \text{ are indep't RV's}}{=} \varphi_X(t)\varphi_{-X}(t) = (1-it)^{-r}(1+it)^{-r}$$

$$= [(1-it)(1+it)]^{-r} = [1-(it)^2]^{-r} = (1+t^2)^{-r}.$$

When $r=1$, we obtain $\frac{1}{1+t^2}$, which is the ch. f. of the double exponential distr^n (145).

(e) If $\int_{-\infty}^{\infty} |\varphi_X(t)|\,dt < \infty$, then can claim that there is a cont's density. Now

$$\int_{-\infty}^{\infty} |\varphi_X(t)\,dt = \int_{-\infty}^{\infty} \frac{dt}{(1+t^2)^r} < \infty \quad \text{if and}$$

only if $2r > 1$, i.e. $r > 1/2$ (recall that $\int_1^\infty \frac{dt}{t^a} < \infty \iff a > 1$ and note that $(1+t^2)^r = t^{2r}\underbrace{(1+t^{-2})^r}_{\to 1} \approx t^{2r}$ as $t \to \infty$)

2. (a) $$\varphi_Y(t) = Ee^{itY_j} \overset{\text{Total Prob'ty Law}}{=} E(e^{itY_j} \mid Y_j = 0)\underbrace{P(Y_j=0)}_{=1-p}$$

$$+ E(e^{itY_j} \mid Y_j > 0)\underbrace{P(Y_j > 0)}_{=p}$$

$$= \underbrace{e^{it\cdot 0}}_{=1} \times (1-p) + p\int_0^\infty e^{itx} \underbrace{\frac{1}{\mu}e^{-x/\mu}}_{\text{density of exp'l distr}^n\text{ with mean }\mu} dx$$

$$= 1 - p + p\frac{1}{1-it\mu}\underbrace{\int_0^\infty e^{-x(\frac{1}{\mu}-it)}\,d\left[x\left(\tfrac{1}{\mu}-it\right)\right]}_{=1\ \text{(as if } \frac{1}{\mu}-it \text{ were real)}}$$

$$= 1 + p\left(\frac{1}{1-it\mu} - 1\right) = 1 + \frac{pit\mu}{1-it\mu}$$

(b) $$\varphi_X(t) \overset{\text{by indep'ce}}{=} (\varphi_Y(t))^n = \Big(\underbrace{1 + \frac{pit\mu}{1-it\mu}}_{\approx \exp\left\{\frac{pit\mu}{1-it\mu}\right\} \text{ as } p \text{ is small}}\Big)^n$$

$$\approx \exp\left\{\frac{npit\mu}{1-it\mu}\right\} \overset{\text{using } np=5,\ \mu=2}{=}$$

$$= \exp\left\{\frac{10it}{1-2it}\right\}.$$

(c) • Not stable as $P(Y=0) = P\left(\bigcap_{j=1}^n \{Y_j = 0\}\right)$
$= (1-p)^n > 0$ (any stable distn has a density!)

• But infitely divisible as for any $m \geq 1$,
$(\varphi_X(t))^{1/m} = \exp\left\{\frac{10m^{-1}it}{1-2it}\right\}$ is again a ch.f.
— can be obtained in the same way as $\varphi_X(t)$, but assuming the number of claims $= \frac{n}{m}$, $n \to \infty$ (in fact, both ch.f.'s correspond to compound Poisson distns).

(d) As noted in (c), $P(X=0)=(1-p)^n$

$= 0.999^{5000} \approx 0.00672$ (or, using $1-p \approx e^{-p}$:

$(1-p)^n \approx e^{-np} = e^{-5} \approx 0.00673$).

3. First need to show: $E|X_n| < \infty$. We have:

$$E|X_n| = E|S_n^2 - n\sigma^2| \le ES_n^2 + n\sigma^2$$

as $|a \pm b| \le |a| + |b|$

$$= \underbrace{\operatorname{Var} S_n}_{= n\sigma^2} + \big(\underbrace{ES_n}_{= nEY_1 = 0}\big)^2 + n\sigma^2 = 2n\sigma^2 < \infty, \text{ OK.}$$

(a) $E[X_{n+1} | \mathcal{F}_n] = E[(S_n + Y_{n+1})^2 - (n+1)\sigma^2 | \mathcal{F}_n]$

$$\stackrel{\boxed{CE1,3}}{=} E[S_n^2 + 2S_nY_{n+1} + Y_{n+1}^2 | \mathcal{F}_n] - (n+1)\sigma^2$$

$$\stackrel{\boxed{CE1}}{=} \underbrace{E[S_n^2 | \mathcal{F}_n]}_{\text{as } S_n^2 = (Y_1 + \dots + Y_n)^2 \text{ is } \mathcal{F}_n\text{-measurable}} + 2\underbrace{E[S_nY_{n+1} | \mathcal{F}_n]}_{\text{same reason}} + \underbrace{E[Y_{n+1}^2 | \mathcal{F}_n]}_{\boxed{CE3},\ \text{indep'ce}} - (n+1)\sigma^2$$

$$\stackrel{\boxed{CE2}}{=} S_n^2 + 2S_n \underbrace{E(Y_{n+1} | \mathcal{F}_n)}_{\boxed{CE3},\ \text{indep.}\ EY_{n+1} = 0} + \underbrace{EY_{n+1}^2}_{= \sigma^2 \text{ (as } EY = 0)} - (n+1)\sigma^2$$

$= S_n^2 - n\sigma^2 = X_n$, OK

(b) Since $\mathcal{F}_n' \subset \mathcal{F}_n$ (as $X_k = S_k^2 - k\sigma^2$, $k=1,\dots,n$,

are all functions of Y_j, $j=1,\dots,n$, there is more information in $(Y_1,\dots,Y_n)$ than in $(X_1,\dots,X_n)$),

$$E(X_{n+1} \mid \mathcal{F}_n') \underset{\boxed{CE4},\ \text{as } \mathcal{F}_n \supset \mathcal{F}_n'}{=} E\Big[\underbrace{E(X_{n+1} \mid \mathcal{F}_n)}^{= X_n \text{ from (a)}} \Big| \mathcal{F}_n'\Big]$$

$$= E(X_n \mid \mathcal{F}_n') \underset{\boxed{CE2}}{=} X_n$$

$\boxed{CE2}$, as X_n is $\mathcal{F}_n'$-measurable.

4. That for all the processes $E|\cdot|<\infty$, is clear. Using $\mathcal{F}_t = \sigma(N_s, s \le t)$:

(a) $$E(N_{t+s} - \lambda(t+s) \mid \mathcal{F}_t) = E(N_t + N_{t+s} - N_t \mid \mathcal{F}_t) - \lambda(t+s) = N_t + \underbrace{E(N_{t+s} - N_t \mid \mathcal{F}_t)}_{= E(N_{t+s} - N_t) = \lambda s} - \lambda(t+s)$$

$= N_t - \lambda t$, OK by indep't increments

Don't forget to say what props of CE's you are using!

(b) Similarly to 3(a).

(c) $$E\left(e^{uN_{t+s} - \lambda(t+s)(e^u-1)} \Big| \mathcal{F}_t\right) = e^{uN_t - \lambda(t+s)(e^u-1)} \times \underbrace{E\left(e^{u(N_{t+s}-N_t)} \Big| \mathcal{F}_t\right)} = e^{uN_t - \lambda t(e^u-1)}, \text{ OK}$$

$$\| \quad E e^{u(N_{t+s}-N_t)} \overset{\text{mgf of } Po(\lambda s)}{=} \sum_{k=0}^{\infty} e^{uk} \frac{(\lambda s)^k}{k!} e^{-\lambda s} = e^{\lambda s(e^u-1)}$$