Tutorial 6 (Solutions)

1. (a) $\{\tau = t\} = \emptyset, \quad t \neq m,$ both events $\in \mathcal{F}_t$,
$= \Omega, \quad t = m,$ of course, so – $\underline{ST}$

(b) $\{\tau \le t\} = \{\tau_1 \wedge \tau_2 \le t\} = \underbrace{\{\tau_1 \le t\}}_{\in \mathcal{F}_t} \cup \underbrace{\{\tau_2 \le t\}}_{\in \mathcal{F}_t}$

so this also $\in \mathcal{F}_t$, OK $\to \underline{ST}$.

(c) $$\{\tau = t\} = \underbrace{\left[\bigcap_{k=0}^{t-1} \underbrace{\left\{\frac{X_{k+1}}{X_k} \le 1\right\}}_{\in \mathcal{F}_k \subset \mathcal{F}_t}\right]}_{\text{so } \in \mathcal{F}_t} \cap \underbrace{\left\{\frac{X_{t+1}}{X_t} > 1\right\}}_{\text{but } \notin \mathcal{F}_t \text{ in general,}}$$

NO GOOD, $\to$ ~~ST~~

(d) $$\{\tau = t\} = \underbrace{\left[\bigcap_{n=0}^{t-1} \underbrace{\left\{\sum_{k=0}^{n} X_k \le X_n^2\right\}}_{\in \mathcal{F}_n \subset \mathcal{F}_t}\right]}_{\text{so } \in \mathcal{F}_t} \cap \underbrace{\left\{\sum_{k=0}^{t} X_k > X_t^2\right\}}_{\in \mathcal{F}_t,}$$

OK $\to \underline{ST}$.

(e) $$\{\tau = t\} = \underbrace{\{X_t > 10\}}_{\in \mathcal{F}_t,} \cap \underbrace{\left[\bigcap_{k=t+1}^{T} \{X_k \le 10\}\right]}_{\text{but } \notin \mathcal{F}_t \text{ in general,}}$$

NO GOOD, $\to$ ~~ST~~

2. (a) As $EY=0$ when $p=P(Y=\pm 1)=1/2$, $\{S_n\}$ is an MG. Using the OST,

$$0 = ES_0 = ES_\tau = a\,\underbrace{P(S_\tau = a)}_{P_a} + b\,P(S_\tau = b)$$
$$= a P_a + b(1-P_a)$$
$$= (a-b)P_a + b, \text{ so } P_a = \frac{b}{b-a} \text{ and}$$

$$P(S_\tau = a) = \frac{b}{b-a}\ ;\ P(S_\tau = b) = \frac{-a}{b-a}.$$

(b) Using the OST, since $Var(Y) = EY^2 = 1$,

$$0 = EX_0 = E(S_\tau^2 - \tau) = ES_\tau^2 - E\tau \underset{(a)}{=}$$
$$= \frac{a^2 b}{b-a} + \frac{b^2(-a)}{b-a} - E\tau = -ab - E\tau, \quad E\tau = -ab.$$

3. (a) $\{Z_n\}$ is clearly adapted to $\mathbb{F}$;

$$E(Z_{n+1} - Z_n \mid \mathcal{F}_n) = E(Y_{n+1}(X_{n+1} - X_n) \mid \mathcal{F}_n)$$
$$= Y_{n+1} \underbrace{E(X_{n+1} - X_n \mid \mathcal{F}_n)}_{=0 \text{ since } \{X_n\} \text{ is an MG}} = 0, \text{ bingo.}$$

$\boxed{CE-2}$ (124) as Y_{n+1} is $\mathcal{F}_n$-measurable

[Well, formally we had to assume that, for each n, $|Y_n| < C_n = \text{const} < \infty$ – and then first verify that $E|Z_n| < \infty$ ...]

(b) $X_n = \xi_1 + \dots + \xi_n$, where ξ_j are i.i.d. RV's, $P(\xi_j = \pm 1) = 1/2$;

$Y_1 = 1$, $Y_n = 2Y_{n-1}$ if $\xi_{n-1} = -1$ (loss);
$Y_n = 1$ if $\xi_{n-1} = 1$ (win).

(c) $P(\tau = n) = P($play 1 lost, play 2 lost, ..., play $(n-1)$ lost, play n won$) = \underset{\sim}{2^{-n}}$ by indep'ce, $\underset{\sim}{n = 1, 2, \dots}$

To find $E\tau$, first compute the p.g.f.

$$g_\tau(z) = \sum_{n=0}^{\infty} z^n P(\tau = n) = \sum_{n=1}^{\infty} \left(\frac{z}{2}\right)^n = \frac{z/2}{1 - z/2} = \frac{z}{2-z};$$

now $$E\tau = g_\tau'(1) = \frac{2}{(2-z)^2}\bigg|_{z=1} = \underset{\sim}{2}.$$

(d) $$E(|Z_n|; \tau > n) = E(2^n - 1; \tau > n)$$

$\Vert \leftarrow$ on this event

$$\underbrace{|-1-2-\dots-2^{n-1}|}_{n \text{ losses!}} = \sum_{k=0}^{n-1} 2^k = \frac{2^n - 1}{2-1} = 2^n - 1$$

$$= (2^n - 1)\underbrace{P(\tau > n)} = (2^n-1)2^{-n} = 1 - 2^{-n} \to 1 \neq 0$$

$$= \sum_{k=n+1}^{\infty} P(\tau = k) = \sum_{k=n+1}^{\infty} 2^{-k} = 2^{-n-1}\underbrace{\sum_{k=0}^{\infty} 2^{-k}}_{=\frac{1}{1-1/2}=2}$$

$$= 2^{-n}$$

Cond'n $\boxtimes$ on slide 182 doesn't hold, no OST.