TUTORIAL 7 (solutions)

1. $\{X_t\}$ is a Gaussian process: for any $0 \le t_1 < t_2 < .. < t_k$,

$$(X_{t_1}, \dots, X_{t_k}) = (t_1 W_{1/t_1}, \dots, t_k W_{1/t_k})$$

is a linear transform of a Gaussian random vector (of the W's) and hence is Gaussian as well.

Therefore by Th^m on sl. 205, we just have to verify that $EX_t = 0$ (which is obvious) and $EX_s X_t = s \wedge t$. As for $0 < s < t$

$$EX_s X_t = E(sW_{1/s} \cdot tW_{1/t}) = st \underbrace{EW_{1/s} W_{1/t}}_{= \frac{1}{s} \wedge \frac{1}{t} = \frac{1}{t}}$$

$$= st \times \frac{1}{t} = s = s \wedge t,$$ done.

2. The RV $X = 2W_{t_1} - W_{t_2}$ is normal - as a linear combination of the components of a normal random vector. So we just have to compute

the mean & var of X:

$$EX = E(2W_{t_1} - W_{t_2}) = 2\underbrace{EW_{t_1}}_{=0} - \underbrace{EW_{t_2}}_{=0} = 0,$$

$$\mathrm{var}(X) = EX^2 = E(2W_{t_1} - W_{t_2})^2 =$$

$$= 4\underbrace{EW_{t_1}^2}_{=t_1} - 4\underbrace{EW_{t_1}W_{t_2}}_{=t_1 \wedge t_2 = t_1} + \underbrace{EW_{t_2}^2}_{=t_2} = t_2,$$

so that $X \sim N(0, t_2)$.

Alternatively, just use the independence of the increments of $\{W_t\}$:

$$X = 2W_{t_1} - (W_{t_1} + (W_{t_2} - W_{t_1})) = \underbrace{W_{t_1}}_{\sim N(0,t_1)} - \underbrace{(W_{t_2} - W_{t_1})}_{\substack{\sim \text{"}-N(0,t_2-t_1)\text{"} \\ = N(0,t_2-t_1) \\ \text{by symmetry}}}$$

indep't RV's

$$\sim N(0, t_1 + (t_2 - t_1)) = N(0, t_2)$$

as the variance of the sum of two indep't RV's = sum of variances, and as the sum of indep't normal RV's is normal again (using ch.f.'s)

3. (a) Need $E(X_{t+h} | \mathcal{F}_t) = X_t$ for any $t \geq 0$, $h > 0$. Now

$$E(X_{t+h} \mid \mathcal{F}_t) = E\left(S_0 e^{(\mu-r)(t+h)+\sigma W_{t+h}} \,\middle|\, \mathcal{F}_t\right)$$

$$\overset{\boxed{CE2}}{=} \underbrace{S_0 e^{(\mu-r)t+\sigma W_t}}_{=X_t} \underbrace{E\left(e^{(\mu-r)h+\sigma(W_{t+h}-W_t)} \,\middle|\, \mathcal{F}_t\right)}_{\boxed{CE3} \text{ as } W_{t+h}-W_t \text{ is indep}^{t} \text{ of } \mathcal{F}_t}$$

as $e^{\sigma W_t}$ is $\mathcal{F}_t$-measurable.

$$= e^{(\mu-r)h} \underbrace{E e^{\sigma(W_{t+h}-W_t)}}_{= e^{\sigma^2 h/2}}$$

as $W_{t+h}-W_t \sim N(0,h)$

$$= X_t \underbrace{e^{(\mu-r)h+\frac{\sigma^2}{2}h}}_{=1}$$

must have $\iff \left(\mu - r + \frac{\sigma^2}{2}\right)h = 0, \ \forall h>0,$ so that $\mu = r - \frac{\sigma^2}{2}$.

(b) $E(S_T-K)^+ = E(S_T - K;\ S_T > K)$

$= E(S_T;\ S_T>K) - E(K;\ S_T>K)$

$= S_0 e^{(r-\frac{\sigma^2}{2})T} E\left(e^{\sigma W_T};\ W_T > w\right) - K P(W_T > w)$

as $S_T > K \iff W_T > \left[\ln(K/S_0) - (r-\frac{\sigma^2}{2})T\right]/\sigma =: w$

$$= S_0 e^{(r-\frac{\sigma^2}{2})T} E\left(e^{\sigma\sqrt{T}Z}; Z > \frac{w}{\sqrt{T}}\right) - K P\left(Z > \frac{w}{\sqrt{T}}\right)$$

as $W_T = \sqrt{T}Z$, $Z \sim N(0,1)$.

Here $P\left(Z > \frac{w}{\sqrt{T}}\right) = P\left(-Z > \frac{w}{\sqrt{T}}\right) = P\left(Z < -\frac{w}{\sqrt{T}}\right)$

as $Z \stackrel{d}{=} -Z$ (symmetric distr^n)

$$= N\left(-\frac{w}{\sqrt{T}}\right) = N\left(\frac{-\ln(K/S_0) + (r - \sigma^2/2)T}{\sigma\sqrt{T}}\right)$$

$$= N\left(\underbrace{\frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}}_{= h \text{ from slide } 79} - \sigma\sqrt{T}\right)$$

$$= N(h - \sigma\sqrt{T}), \quad \text{and}$$

$$E\left(e^{\sigma\sqrt{T}Z}; Z > \frac{w}{\sqrt{T}}\right) = \frac{1}{\sqrt{2\pi}} \int_{w/\sqrt{T}}^{\infty} e^{\sigma\sqrt{T}x - x^2/2}\, dx$$

as $-\frac{1}{2}(x^2 - 2\sigma\sqrt{T}x) = -\frac{1}{2}(x - \sigma\sqrt{T})^2 + \frac{\sigma^2 T}{2}$

$$= \frac{e^{\sigma^2 T/2}}{\sqrt{2\pi}} \int_{w/\sqrt{T}}^{\infty} e^{-(x-\sigma\sqrt{T})^2/2}\, dx \underset{y = x - \sigma\sqrt{T}}{=} \frac{e^{\sigma^2 T/2}}{\sqrt{2\pi}} \int_{\underbrace{\frac{w}{\sqrt{T}} - \sigma\sqrt{T}}_{=-h}}^{\infty} e^{-y^2/2}\, dy$$

$$= e^{\frac{\sigma^2 T}{2}} P(Z > -h) = e^{\frac{\sigma^2 T}{2}} N(h) \quad \text{(as above)},$$

so that

$$E(S_T - K)^+ = S_0 e^{rT} N(h) - K N(h - \sigma\sqrt{T}).$$

Cf. with the BS-formula for C on slide 76

4. Using the MG $X_t = W_t^2 - t$ & OST:

$$0 = EX_0 = EX_\tau = E(\underbrace{W_\tau^2}_{} - \tau) = E(a + b\tau - \tau)$$

$= (\sqrt{a+b\tau})^2$ when crossing u_t first;

$= (-\sqrt{a+b\tau})^2 = (a+b\tau)$ when crossing v_t first.

$= a - (1-b)E\tau$,

so that

$$E\tau = \frac{a}{1-b}.$$

Showing that $\{W_t^2 - t\}$ is an MG:

$$E(W_t^2 - t \mid \mathcal{F}_s) = E((W_s + (W_t - W_s))^2 - t \mid \mathcal{F}_s)$$

$$= \underbrace{E(W_s^2 \mid \mathcal{F}_s)}_{= W_s^2} + \underbrace{E((W_t - W_s)^2 \mid \mathcal{F}_s)}_{= E(W_t - W_s)^2 = t - s} + \underbrace{2E(W_s(W_t - W_s) \mid \mathcal{F}_s)}_{= W_s E(W_t - W_s \mid \mathcal{F}_s) = W_s \underbrace{E(W_t - W_s)}_{=0} = 0}$$

as W_s^2 is $\mathcal{F}_s$-meas.

as $W_t - W_s$ is indep^t of $\mathcal{F}_s$

as W_s is $\mathcal{F}_s$-m.

as $W_t - W_s$ is indep^t of $\mathcal{F}_s$

$$- t =$$

$$= W_s^2 + (t - s) + 0 - t = W_s^2 - s = X_s, \quad MG!$$