# TUTORIAL 9 SOLUTIONS

**1.** $$\operatorname{cov}(I_t(f), I_t(g)) \overset{\uparrow}{=} E\, I_t(f) I_t(g)$$

since $E I_t(f) = E I_t(g) \overset{\boxed{II3}}{=} 0$

assuming for the moment that $t=T$; same argument works for $t<T$

$$\overset{\downarrow}{=} E\left[\sum_{k=1}^{n} X_k (W_{t_k} - W_{t_{k-1}}) \times \sum_{j=1}^{n} Y_j (W_{t_j} - W_{t_{j-1}})\right]$$

$$\overset{\uparrow}{=} \sum_{k=1}^{n} E\left(X_k Y_k (W_{t_k} - W_{t_{k-1}})^2\right) + \sum_{k\neq j}\sum E\left[X_k X_j (W_{t_k} - W_{t_{k-1}})(W_{t_j} - W_{t_{j-1}})\right]$$

using $\sum_{k=1}^{n} a_k \times \sum_{j=1}^{n} b_j = \sum_{k=1}^{n} a_k b_k + \sum_{k\neq j}\sum a_k b_j$

Here

$$E\left(X_k Y_k (W_{t_k} - W_{t_{k-1}})^2\right) \overset{\boxed{CE4}}{=} E\, E\left(-\shortparallel- \mid \mathcal{F}_{t_{k-1}}\right)$$

$$\overset{\uparrow}{=} E\Big[X_k Y_k \underbrace{E\left((W_{t_k} - W_{t_{k-1}})^2 \mid \mathcal{F}_{t_{k-1}}\right)}_{\overset{\uparrow}{=} E(W_{t_k} - W_{t_{k-1}})^2 = t_k - t_{k-1}}\Big]$$

$\boxed{CE2}$, since $X_k, Y_k$ are $\mathcal{F}_{t_{k-1}}$-measurable

$\boxed{CE3}$, since $W_{t_k} - W_{t_{k-1}}$ doesn't depend on $\mathcal{F}_{t_{k-1}}$

$$= (t_k - t_{k-1}) \underbrace{E(X_k Y_k)}_{= E(f_t g_t)} \text{ when } t \in [t_{k-1}, t_k).$$

Therefore the first sum equals

$$\sum_{k=1}^{n}(t_k - t_{k-1})E\left(f_{t_{k-1}}\, g_{t_{k-1}}\right) = \int_0^t E(f_s g_s)\,ds$$

since $E(f_s g_s)$ is piece-wise constant (on $[t_{k-1}, t_k)$)

As for the second (double) sum, it's zero: say, for $j<k$,

$$E\left(X_k Y_j (W_{t_k} - W_{t_{k-1}})(W_{t_j} - W_{t_{j-1}})\right) \overset{\boxed{CE4}}{=} EE\left(-\shortparallel- \mid \mathcal{F}_{t_{k-1}}\right)$$

$$\overset{\boxed{CE2}}{=} E\left[X_k \underbrace{Y_j (W_{t_j} - W_{t_{j-1}})}_{\mathcal{F}_{t_j}\text{-m., but } \mathcal{F}_{t_j} \subset \mathcal{F}_{t_{k-1}}\ (j<k!!)} E\left(W_{t_k} - W_{t_{k-1}} \mid \mathcal{F}_{t_{k-1}}\right)\right]$$

$X_k$: $\mathcal{F}_{t_{k-1}}$-m.

$= 0$ since $E\left(W_{t_k} - W_{t_{k-1}} \mid \mathcal{F}_{t_{k-1}}\right) \overset{\boxed{CE3}}{=} E\left(W_{t_k} - W_{t_{k-1}}\right) = 0$

by indep'ce of $W_{t_k} - W_{t_{k-1}}$ from $\mathcal{F}_{t_{k-1}}$

2. $Y_t = f(X_t)$, where $f(x) = e^x$ with $f'(x) = f''(x) = e^x = f(x)$ and $dX_t = g_t\,dW_t - \frac{1}{2}g_t^2\,dt$

By Itô's formula,

$$dY_t = f'(X_t)\,dX_t + \frac{1}{2} f''(X_t)(dX_t)^2$$

$$= f(X_t)\left(g_t\,dW_t - \frac{1}{2}g_t^2\,dt\right) + \frac{1}{2} f(X_t)\,g_t^2\,dt$$

as $(dX_t)^2 = g_t^2 (dW_t)^2 = g_t^2\,dt$

$$= f(X_t)\,g_t\,dW_t \iff Y_t = Y_0 + \underbrace{\int_0^t e^{X_t} g_t\,dW_t}_{\text{MG by } \boxed{II3}}$$

Bingo.

3. (a) First note that $S_0 = c = 5$. Next note that

$$S_t = f(t, W_t) \text{ with } f(t,x) = 5e^{at+bx},$$

so that $f_t' = af$, $f_x' = bf$, $f_{xx}'' = b^2 f$.

Hence by Itô's formula,

$$dS_t = f_t'(t, W_t)\,dt + f_x'(t, W_t)\,dW_t + \frac{f_{xx}''(t,W_t)}{2}\underbrace{(dW_t)^2}_{=dt}$$

$$= \Big[a\underbrace{f(t,W_t)}_{S_t} + \frac{b^2}{2}\underbrace{f(t,W_t)}_{S_t}\Big]dt + b\underbrace{f(t,W_t)}_{S_t}\,dW_t$$

$$= \left(a + \frac{b^2}{2}\right)S_t\,dt + bS_t\,dW_t = 0.2S_t\,dt + S_t\,dW_t.$$

the SDE!

Hence $S_t$ will satisfy the SDE if

$$\begin{cases} a + \frac{b^2}{2} = 0.2 \\ b = 1 \end{cases} \iff \underbrace{a = -0.3}, \ \underbrace{b = 1}; \ \underbrace{c = 5}.$$

found this before!

(b) $X_t = 1/S_t = 0.2\left(e^{-0.3t + W_t}\right)^{-1} = 0.2 e^{0.3t - W_t}\Big|_{t=0} = \underbrace{0.2}$ $\Updownarrow$ $X_0$!

Now by Itô's formula, as above,

$$dX_t = 0.3 X_t dt - X_t dW_t + \frac{1}{2} X_t dt$$
$$= 0.8 X_t dt - X_t dW_t, \text{ bingo.}$$

4. $X_t = f(W_t)$ with $f(x) = \cos x$; since $f'(x) = -\sin x$, $f''(x) = -\cos x$, by Itô's formula

$$dX_t = f'(W_t) dW_t + \frac{f''(W_t)}{2} (dW_t)^2$$
$$= \underbrace{-\sin(W_t) dW_t - \frac{1}{2} \cos(W_t) dt}.$$