Tutorial 10: Solutions

1. (a) $X_t = X_s + (t-s) + 2(W_t - W_s)$;

given $X_s = x$, the cond'l distr^n of this is $N(x + t - s,\ 4(t-s))$, so

$$p(s,x;t,y) = \frac{1}{2\sqrt{2\pi(t-s)}} \exp\left\{-\frac{(y-x-t+s)^2}{8(t-s)}\right\}.$$

(b) As $\mu \equiv 1$, $\sigma \equiv 2$,

BWKE for $v = v(s,x)$: $v'_s = -v'_x - 2v''_{xx}$.

FWKE for $u = u(t,y)$: $u'_t = -u'_y + 2u''_{yy}$.

(c)
$$u(t,y) = \frac{1}{2\sqrt{2\pi t}} e^{-(y-x_0-t)^2/8t}$$
$$= \varphi(8t,\ y - x_0 - t)$$

with
$$\varphi(s,z) = \pi^{-1/2} s^{-1/2} e^{-z^2/s}.$$

Firstly, compute:

$$\varphi'_s = -\frac{1}{2s}\varphi + \frac{z^2}{s^2}\varphi,$$

$$\varphi'_z = -\frac{2z}{s}\varphi, \quad \varphi''_{zz} = -\frac{2}{s}\varphi + \frac{4z^2}{s^2}\varphi.$$

Now we can use this to differentiate the original $u(t,y)$:

$$u'_t = \frac{\partial u}{\partial t} = \frac{\partial \varphi}{\partial s} \times \frac{\partial s}{\partial t} + \frac{\partial \varphi}{\partial z} \times \frac{\partial z}{\partial t}$$
$$= \left[-\frac{1}{2s} + \frac{z^2}{s^2}\right]\varphi \times 8 + \left(-\frac{2z}{s}\varphi\right) \times (-1)$$
$$= -\frac{4}{s}\varphi + \frac{8z^2}{s^2}\varphi + \frac{2z}{s}\varphi;$$

$$u'_y = \frac{\partial \varphi}{\partial s} \times \underbrace{\frac{\partial s}{\partial y}}_{=0} + \frac{\partial \varphi}{\partial z} \times \underbrace{\frac{\partial z}{\partial y}}_{=1} = -\frac{2z}{s}\varphi;$$

$$u''_{yy} = \varphi''_{zz} = -\frac{2}{s}\varphi + \frac{4z^2}{s^2}\varphi.$$

Now just substitute these results into the FWKE: e/thing will cancel out.

2. (a) As $\mu(t,y) = -\alpha y$
($\Leftrightarrow$ $\mu(s,x) = -\alpha x$, of course!)

and $\sigma(t,y) \equiv \sigma = \text{const}$,

BWKE: $v'_s = \alpha x v'_x - \frac{\sigma^2}{2} v''_{xx}$,

FWKE: $u'_t = -(-\alpha y u)'_y + \frac{\sigma^2}{2} u''_{yy}$
$$= \alpha u + \alpha y u'_y + \frac{\sigma^2}{2} u''_{yy}.$$

(b) $dY_t = d(e^{-\alpha t} Z_{f(t)})$ $\left[f(t) = \frac{\sigma^2(e^{2\alpha t}-1)}{2\alpha}\right]$

as $e^{-\alpha t}$ is a deterministic f^n

$$= -\alpha e^{-\alpha t} Z_{f(t)}\,dt + e^{-\alpha t} dZ_{f(t)}$$

$$\boxed{\text{as } f'(t)=\sigma^2 e^{2\alpha t}} \quad = -\alpha Y_t\,dt + e^{-\alpha t}\sqrt{\sigma^2 e^{2\alpha t}}\,dW^*_t$$

$$= -\alpha Y_t\,dt + \sigma\,dB^*_t\,.$$

(c) The distr^n of $e^{-\alpha t} Z_{f(t)} = e^{-\alpha t}(x + W_{f(t)})$ is $N(xe^{-\alpha t}, \underbrace{e^{-2\alpha t} f(t)}_{=\frac{\sigma^2}{2\alpha}(1-e^{-2\alpha t})})$

so the density of Y_t is

$$\frac{1}{\sqrt{\pi\sigma^2(1-\underbrace{e^{-2\alpha t}}_{\to 0})/\alpha}} \exp\left\{-\frac{(y-\overbrace{xe^{-\alpha t}}^{\to 0})^2}{\sigma^2(1-\underbrace{e^{-2\alpha t}}_{\to 0})/\alpha}\right\}$$

$$\to \frac{1}{\sqrt{2\pi\sigma^2_\infty}} e^{-y^2/2\sigma^2_\infty}, \quad \sigma^2_\infty = \sigma^2/2\alpha,$$

the density of $N(0, \sigma^2_\infty)$.

So, for any initial value Y_0, the distr^n of Y_t converges to $N(0,\sigma^2_\infty)$ as $t\to\infty$. That is, the process $\{Y_t\}$ is <u>ergodic</u>, with this normal law being its stationary distr^n.

(d) $$0 = -\frac{\partial}{\partial y}\left(\mu(y)\pi(y)\right) + \frac{1}{2}\frac{\partial^2}{\partial y^2}\left(\sigma^2(y)\pi(y)\right)$$

$$= \frac{d}{dy}\left[\alpha y\,\pi(y) + \frac{\sigma^2}{2}\frac{d}{dy}\pi(y)\right],$$

So that

$[\cdots] = .G = 0$ by the "zero rule"
(see slides 306a)b)):

$$0 = \alpha y \pi(y) + \frac{\sigma^2}{2}\pi'(y),$$

$$\frac{\pi'}{\pi} = -\frac{2\alpha y}{\sigma^2} \to \ln \pi = -\frac{\alpha y^2}{\sigma^2} + C_0,$$

$$\pi(y) = e^{C_0} e^{-y^2/2(\sigma^2/2\alpha)} \leftrightarrow \underline{N(0, \sigma^2/2\alpha)}$$

we obtained the same distribution as the limiting one from part (c): the density of Y_t does converge to the stationary one as $t \to \infty$.