(a) \(X_t = X_s + (t-s) + 2(W_t-W_s)\); given \(X_s=x\), the conditional distribution of this is \(N(x+t-s, 4(t-s))\), so

\[
p(s,x; t,y) = \frac{1}{2 \sqrt{2\pi} (t-s)} \exp \left\{ -\frac{(y-x-t+s)^2}{8(t-s)} \right\}.
\]

(b) As \(\mu=1\), \(\sigma=2\),

\begin{align*}
\text{BKE for } v &= v(s,x): \quad v'_s = -v'_x - 2v''_x, \\
\text{FOM KE for } u &= u(t,y): \quad u'_t = -u'_y + 2u''_y.
\end{align*}

(c) \(u(t,y) = \frac{1}{2 \sqrt{2\pi t}} e^{-(y-x_0-t)^2/8t}\)

\[
= \psi(8t, y-x_0-t)
\]

with

\[
\psi(s, z) = \frac{1}{\pi} \frac{1}{\sqrt{s}} e^{-z^2/s}.
\]

Firstly, compute:

\[
\begin{align*}
\psi'_s &= -\frac{1}{2s} \psi + \frac{z^2}{s^2} \psi, \\
\psi'_z &= -\frac{2z}{s} \psi, \quad \psi''_s = -\frac{2}{s} \psi + \frac{4z^2}{s^2} \psi.
\end{align*}
\]

Now we can use this to differentiate the original \(u(t,y)\):
\( u' \)_t = \frac{\partial u}{\partial t} = \frac{\partial P}{\partial s} \cdot \frac{\partial s}{\partial t} + \frac{\partial P}{\partial z} \cdot \frac{\partial z}{\partial t} \\
= [-\frac{1}{2s} + \frac{2z^2}{s^2}]P \times 8 + (-\frac{2z}{s}) \times (-1) \\
= -\frac{4}{s}P + \frac{8z^2}{s^2}P + \frac{2z}{s}P; \\

u'_{yy} = \frac{\partial P}{\partial y} \times \frac{\partial s}{\partial y} + \frac{\partial P}{\partial z} \times \frac{\partial z}{\partial y} = -\frac{2z}{s}P; \\

u''_{yy} = u''_{zz} = -\frac{2z}{s}P + \frac{4z^2}{s^2}P.

Now just substitute these results into the FWKE: e^{f(t)} will cancel out.

2. (a) As \( \mu(t, y) = -\frac{dy}{dt} \) \\
(\Leftrightarrow \mu(s, x) = -dx, of course!) \\

and \( g(t, y) = G = \text{const} \),

**BWKE:** \( v'_s = dx v'_x - \frac{6^2}{2} v''_{xx} \),

**FWKE:** \( u'_t = -(-dyu)'_y + \frac{6^2}{2} u''_{yy} \\
= du + dy u'_y + \frac{6^2}{2} u''_{yy}. \)

(B) \( \frac{dY}{dt} = d(e^{-dt} Z_{f(t)}) \) \[ f(t) = \frac{e^{2dt}}{2d} \]

(as \( e^{-dt} \) is a deterministic \( \mathbb{E} \))

\[ = -d e^{dt} Z_{f(t)} dt + e^{dt} dZ_{f(t)}. \]
\[ \text{as } f(t) = e^{2t} \]

\[ = -dY_t \, dt + e^{-dt} \sqrt{\sigma^2 e^{2dt}} \, dB_t^* \]

\[ = -dY_t \, dt + \sigma dB_t^* \]

(C) The distrn of \( e^{-dt} Z_{f(t)} = e^{-dt} (x + W_{f(t)}) \)

is \( N(x e^{-dt}, e^{-2dt} f(t)^2) \)

\[ \frac{\sigma^2}{2d} \left( 1 - e^{-2dt} \right) \]

so the density of \( Y_t \) is

\[ \frac{1}{\sqrt{2\pi \sigma^2}} \exp \left\{ -\frac{(y - x e^{-dt})^2}{2\sigma^2 (1 - e^{-2dt})} \right\} \]

\[ \to 0 \]

\[ \to \frac{1}{\sqrt{2\pi \sigma^2}} e^{-y^2/2\sigma^2} \]

\[ \sigma^2 = \sigma^2_\infty \]

the density of \( N(0, \sigma^2_\infty) \).

So, for any initial value \( Y_0 \),

the distrn of \( Y_t \) converges to \( N(0, \sigma^2_\infty) \)

as \( t \to \infty \). That is, the process

\( \{Y_t\} \) is ergodic, with this normal law being its stationary distrn.

(d) \[ \theta = -\frac{1}{\sigma^2} (\mu(y) \pi(y)) + \frac{1}{2} \frac{\partial^2}{\partial y^2} (\theta^2) \pi(y) \]

\[ = \frac{d}{dy} \left[ \phi(y) \pi(y) + \frac{\sigma^2}{2} \frac{d}{dy} \pi(y) \right] \]
So that

\[ \ldots \mid = C = 0 \] by the "zero rule"

(see slides \(306\choose68\)):

\[ 0 = \frac{d}{dy} \Pi(y) + \frac{d^2}{2} \Pi'(y), \]

\[ \frac{\Pi'}{\Pi} = -\frac{2dy}{\sigma^2} \to \ln \Pi = -\frac{dy^2}{\sigma^2} + C_0, \]

\[ \Pi(y) = e^{C_0} e^{-\frac{y^2}{2}(\frac{1}{\sigma^2}/2d)} \leftrightarrow \mathcal{N}(0, \frac{\sigma^2}{2d}), \]

we obtained the same distribution as the limiting one from part (c):

the density of \(Y_t\) does converge to the stationary one as \(t \to \infty\).