TUTORIAL 11 : SOLUTIONS

1. (a) $\mu(x) = \frac{1}{2}x$, $\sigma \equiv 1$, so

BWKE: for $v = v(s,x)$,

$$v'_s = -\frac{x}{2}v'_x - \frac{1}{2}v''_{xx}$$

FWKE: for $u = u(t,y)$,

$$u'_t = -\left(\frac{y}{2}u\right)'_y + \left(\frac{1}{2}u\right)''_{yy}$$

$$u'_t = -\frac{1}{2}u - \frac{y}{2}u'_y + \frac{1}{2}u''_{yy}$$

(b) Take $\psi(t,a) \equiv 0$, $\psi(t,b) \equiv 1$, $t>0$;

the f^n $v(s,x) := E_{s,x}\psi(\tau, X_\tau) =$

$= P_{s,x}(X_\tau = b) = P_{0,x}(X_\tau = b) =: V(x)$

solves the BWKE: as $v'_s = 0$, $v'_x = V'$, $v''_{xx} = V''$, it becomes:

$$0 = -\mu V' - \frac{\sigma^2}{2}V'' \quad \text{with } V(a)=0, V(b)=1.$$

$$= -\frac{x}{2}V' - \frac{1}{2}V''.$$

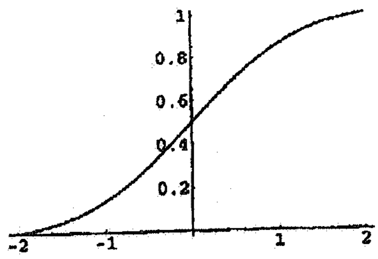
Solving: $(\ln V')' = V''/V' = -x$, so

$\ln V' = -\frac{1}{2}x^2 + C_0$, or $V' = e^{C_0}e^{-x^2/2}$, so that

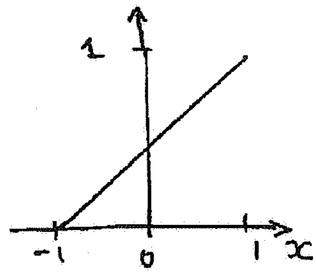
$V(x) = C_1 N(x) + C_2$, where $N(x)$ is the std normal distribution function. Using the boundary conditions we get:

$$V(x) = \frac{N(x) - N(a)}{N(b) - N(a)}, \quad x \in [a,b].$$

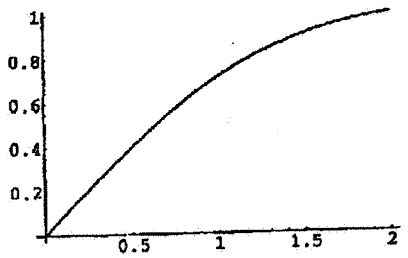
(c) [i]: (-2,2)



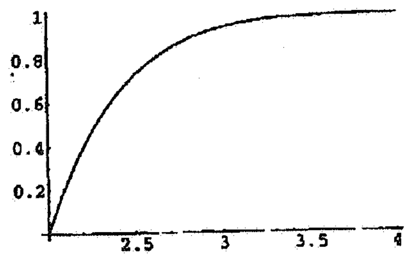
[ii]: (-1,1)



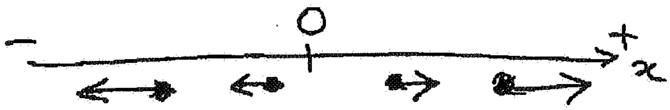
[iii]: (0,2)



[iv]: (2,4)



The drift coeff't $\mu(x) = x/2$ can be interpreted as a display of a "force" pushing our process away from $x = 0$:



For a small & "centred" (a, b) (e.g. $(-1, 1)$), the influence of the force is negligible, so $V(x)$ is almost the same as in the case of std BM process. For a larger interval (e.g. $(-2,2)$), there are rather "flat" pieces near the end points: if $X_0 = x$

is close to 2, we'll hit 2 with a very high prob'ty due to the force! For "skew" intervals, of a fixed length, the greater the distance from 0, the higher the prob'ty to hit the "exterior" boundary as the $\underset{=(\text{drift term})}{\text{force}}$ becomes .

(d) $$P(\max_{t\geq 0} X_t > b \,/\, X_0 = x)$$

$$= \lim_{a\to -\infty} V(x) = \frac{N(x)}{N(b)}, \quad x \leq b$$

$(= 1$ if $x > b)$.

2. (a) $\mu(x) = -\gamma_1 x + \gamma_2(1-x)$, $\sigma^2(x) = x(1-x)$, $x \in (0,1)$,

so:

BWKE: for $v = v(s,x)$,

$$v'_s = (\gamma_1 x - \gamma_2(1-x))v'_x - \frac{1}{2}x(1-x)v''_{xx}.$$

FWKE: for $u = u(t,y)$,

$$u'_t = ((\gamma_1 y - \gamma_2(1-y))u)'_y + \frac{1}{2}(y(1-y)u)''_{yy}$$

$$= (\gamma_1 + \gamma_2)u + (\gamma_1 y - \gamma_2(1-y))u'_y$$

$$+ \frac{1}{2}[-2u + 2(1-2y)u'_y + y(1-y)u''_{yy}], \text{ i.e.}$$

$$u'_t = (\gamma_1 + \gamma_2 - 1)u + (1 - \gamma_2 + (\gamma_1 + \gamma_2 - 2)y)u'_y + \frac{y(1-y)}{2}u''_{yy}.$$

(b) From the FWKE:

$$0 = \frac{d}{dy}\left[-\mu(y)\pi(y) + \frac{d}{dy}\frac{\sigma^2(y)}{2}\pi(y)\right],$$

and from "zero rule" we know that $[\cdots]=0$, so have to solve:

$$0 = -\mu\pi + \frac{d}{dy}\left(\frac{\sigma^2(y)}{2}\pi(y)\right) =$$

$$= -(1-2y)\pi(y) + \frac{1}{2}\left(y(1-y)\pi(y)\right)' =$$

$$= -\frac{1}{2}(1-2y)\pi(y) + \frac{1}{2}y(1-y)\pi'(y)$$

$$= \pi' g - \pi g \quad \text{with } g(y) = \frac{1}{2}y(1-y)$$

$$= g^2\left(\frac{\pi}{g}\right)', \text{ i.e. } \left(\frac{\pi}{g}\right)' = 0, \text{ which means}$$

that $\pi = C_0 g = Cy(1-y)$. Using

$$1 = \int_0^1 \pi(y)\,dy = C\int_0^1 y(1-y)\,dy = C\left[\frac{y^2}{2} - \frac{y^3}{3}\right]_0^1 = \frac{C}{6},$$

we get $\pi(y) = 6y(1-y),\ y \in (0,1)$.