Assignment 1

1. Let \((X, d)\) be a metric space.
   
   (a) Show, directly from the definition, that if \(x \in X\) then \(\{x\}\) is closed in \(X\).
   
   (b) Deduce that every finite subset of \(X\) is closed in \(X\).
   
   (c) Deduce that every subset of \(X\) with finite complement is open in \(X\).
   
   (d) Define a collection of open sets on the positive integers \(\mathbb{N}\) by \(U \subseteq X\) is open if \(U\) has a finite complement. Prove that this defines a topology on \(\mathbb{N}\), i.e satisfies the three properties for open sets, but this cannot be the topology of open sets for any metric on \(\mathbb{N}\). (Hint: Consider the disjointness properties of open balls, which must be in the topology).

2. Let \(A\) and \(B\) be bounded subsets of a metric space \((X, d)\) such that \(A \cap B \neq \emptyset\). Show that
   
   \[
   \text{diam}(A \cup B) \leq \text{diam}(A) + \text{diam}(B).
   \]
   
   What can you say if \(A\) and \(B\) are disjoint?

3. Let \((X, d_X)\) and \((Y, d_Y)\) be metric spaces and let \((X \times Y, d)\) be the product metric space. Let \(\text{int}A\) denote the interior of a subset of a metric space. Show that if \(A \subseteq X\) and \(B \subseteq Y\), then
   
   \[
   \text{int}A \times \text{int}B = \text{int}(A \times B).
   \]

4. Let \((X, d)\) be a metric space and let \(A\) be a non-empty subset of \(X\). Recall that for each \(x \in X\), the distance from \(x\) to \(A\) is
   
   \[
   d(x, A) = \inf\{d(x, a) : a \in A\}.
   \]
   
   (a) Prove that \(\overline{A} = \{x \in X : d(x, A) = 0\}\).
   
   (b) Prove that \(|d(x, A) - d(y, A)| \leq d(x, y)|\) for all \(x, y \in X\). [Hint: first show that \(d(x, A) \leq d(x, y) + d(y, A)\).]
   
   (c) Deduce the function \(f : X \rightarrow \mathbb{R}\) defined by \(f(x) = d(x, A)\) is continuous.
   
   (d) Show that if \(x \notin \overline{A}\) then \(U = \{y \in X : d(y, A) < d(x, A)\}\) is an open set in \(X\) such that \(\overline{A} \subset U\) and \(x \notin U\).

5. Determine whether the following sequences of functions converge uniformly.
   
   (a) \(f_n(x) = e^{-nx^2}, \quad x \in [0, 1]\);
   
   (b) \(g_n(x) = e^{-x^2/n}, \quad x \in [0, 1]\).

6. Consider the sequence \(f_n(x) = \cos(x/n)\) in \(C[-1, 1]\) equipped with the metric \(d_\infty(f, g) = \sup\{|f(x) - g(x)| : x \in [-1, 1]\}\). Show that the sequence \((f_n)\) is Cauchy in \((C[-1, 1], d_\infty)\). Prove that the sequence has no limit and hence the space is not complete.
7 Let $X$ be a nonempty set and let $(Y, d)$ be a complete metric space. Let $f : X \to Y$ be an injective function and define

$$d_f(x, y) = d(f(x), f(y))$$

for $x, y \in X$.

(a) Explain briefly why $d_f$ is a metric on $X$.
(b) Show that $(X, d_f)$ is a complete metric space if $f(X)$ is a closed subset of $Y$.

8. Let $a > 0$, and let

$$f(x) = \frac{1}{2} \left( x + \frac{a}{x} \right) \text{ for } x > 0.$$ 

(a) Show that $f(x) \geq \sqrt{a}$ for all $x > 0$. Hence $f$ defines a function $f : X \to X$ where $X = [\sqrt{a}, \infty)$.
(b) Show that $f$ is a contraction mapping when $X$ is given the usual metric.
(c) Fix $x_0 > \sqrt{a}$ and $x_{n+1} = f(x_n)$ for all $n \geq 0$. Show that the sequence $\{x_n\}$ converges and find its limit with respect to the usual metric on $\mathbb{R}$.

9 Let $X$ be a complete normed vector space over $\mathbb{R}$. (Recall a norm $\| \cdot \|$ satisfies $\|a + b\| \leq \|a\| + \|b\|$, $\|a\| \geq 0$ with $\|a\| = 0$ if and only if $a = 0$ and $\|\lambda a\| = |\lambda|\|a\|$. We then define a metric by $d(a, b) = \|a - b\|$.) A **sphere** in $X$ is a set

$$S(a, r) = \{ x \in X : d(x, a) = \| x - a \| = r \}$$

where $a \in X$ and $r > 0$.

(a) Show that each sphere in $X$ is nowhere dense.
(b) Show that there is no sequence of spheres $\{S_n\}$ in $X$ whose union is $X$.
(c) Give a geometric interpretation of the result in (b) when $X = \mathbb{R}^2$ with the Euclidean norm.

10. Let $(X, d)$ and $Y, d'$ be metric spaces and let $f, g : X \to Y$ be continuous.

(a) Show that the set $\{ x \in X : f(x) = g(x) \}$ is a closed subset of $X$.
(b) Show that $f, g : X \to \mathbb{R}$ are continuous then $f - g$ is continuous and $\{ x \in X : f(x) < g(x) \}$ is open.