1. Let $A = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ and $B = \{(x,y) \in \mathbb{R}^2 : (x-2)^2 + y^2 < 1\}$. Determine whether $X = A \cup B$, $Y = \overline{A} \cup \overline{B}$ and $Z = \overline{A} \cup B$ are connected subsets of $\mathbb{R}^2$ with the usual topology. Explain your answers briefly.

**Solution:** (a) $A$ and $B$ are non-empty open subsets of $\mathbb{R}^2$, hence non-empty open subsets of $X$, with $A \cup B = X$ and $A \cap B = \emptyset$. Hence $X = A \cup B$ is disconnected.
(b) $\overline{A}$, $\overline{B}$ are convex subsets of $\mathbb{R}^2$, hence path connected, hence connected. Since $A \cap B = \{(0,0)\} \neq \emptyset$, it follows that $Y = \overline{A} \cup \overline{B}$ is connected.
(c) $Z = \overline{A} \cup B$ contains the interval $C = [-1,3] \times \{0\}$ which intersects the connected sets $\overline{A}$ and $B$. Hence $\overline{A} \cup C$, $B \cup C$ and $Z = (\overline{A} \cup C) \cup (B \cup C)$ are connected.

2. Let $X$ be a connected topological space and let $f : X \rightarrow \mathbb{R}$ be a continuous function, where $\mathbb{R}$ has the usual topology. Show that if $f$ takes only rational values, i.e. $f(X) \subset \mathbb{Q}$, then $f$ is a constant function.

**Solution:** Assume $f$ is not constant, say $f(x) = a < f(y) = b$ for some $x, y \in X$. Then $f(X)$ is a connected subset of $\mathbb{R}$, so is an interval containing $[a,b]$. But thiscontains irrationals (e.g. $a + \frac{1}{\sqrt{2}}(b-a)$), contradicting the assumption that $f(X) \subset \mathbb{Q}$.

3. Prove that $X = \{(x,y) \in \mathbb{R}^2 : xy = 0\}$ is not homeomorphic to $\mathbb{R}$ (with the usual topologies). [Hint: consider the effect of removing points from $X$ and $\mathbb{R}$.] 

**Solution:** Removing any point $x$ from $\mathbb{R}$ gives two connected components for $\mathbb{R}\backslash\{x\}$, namely $(-\infty,x)$ and $(x,\infty)$. However, removing the origin $\{(0,0)\}$ from $X$ gives 4 connected components: the positive $x$-axis, the negative $x$-axis, the positive $y$-axis and the negative $y$-axis. But any homeomorphism $f : X \rightarrow \mathbb{R}$ restricts to a homeomorphism $g$ from $X \backslash \{(0,0)\}$ to $\mathbb{R} \backslash f(0,0)$, and $g$ must preserve connected components – contradiction.

4. Prove that if $X$ and $Y$ are path connected, then $X \times Y$ is also path connected.

**Solution:** Let $(x_0,y_0), (x_1,y_1)$ be two points in $X \times Y$. Since $X$ is path connected there is a continuous function $\alpha : [0,1] \rightarrow X$ such that $\alpha(0) = x_0$ and $\alpha(1) = x_1$. Since $Y$ is path connected, there is a continuous function $\beta : [0,1] \rightarrow X$ such that $\beta(0) = y_0$ and $\beta(1) = y_1$. Combining these gives a function $\gamma = (\alpha, \beta) : [0,1] \rightarrow X \times Y$, $\gamma(t) = (\alpha(t), \beta(t))$, with $\gamma(0) = (x_0,y_0)$ and $\gamma(1) = (x_1,y_1)$. Further, $\gamma$ is continuous since each component is continuous. Hence $X \times Y$ is path connected.

5. Let $(X, T)$ be a Hausdorff topological space, and let $T'$ be another topology on $X$ with $T' \supseteq T$ and $T' \neq T$. Prove that $(X, T')$ is not compact. [Hint: consider the identity map $f : X \rightarrow X$ with $f(x) = x$.]

**Solution:** The identity $f : (X, T') \rightarrow (X, T)$ is continuous since $T' \supseteq T$ and is a bijection. If $(X, T')$ is compact and $(X, T)$ Hausdorff, then $f$ would be a homeomorphism. Hence $f^{-1} : (X, T) \rightarrow (X, T')$ would be continuous. But $T'$ contains an open set $U$ which is not in $T$, and $(f^{-1})^{-1}(U) = f(U) = U$ is not in $T$, contradicting the continuity of $f^{-1}$. 

---

**Solutions to Assignment 4.**