620-311 Metric Spaces: Problem Set 1

1. Check if the following functions are metrics on X.

(a) $d(x, y) = |x^2 - y^2|$ for $x, y \in X = \mathbb{R}$

(b) $d(x, y) = |x^2 - y^2|$ for $x, y \in X = (-\infty, 0]$

(c) $d(x, y) = |\arctan x - \arctan y|$ for $x, y \in X = \mathbb{R}$

2. (**French railroad metric**) Let $X = \mathbb{R}^2$ and let d be the usual metric. Denote by $\mathbf{0} = (0, 0)$ and define

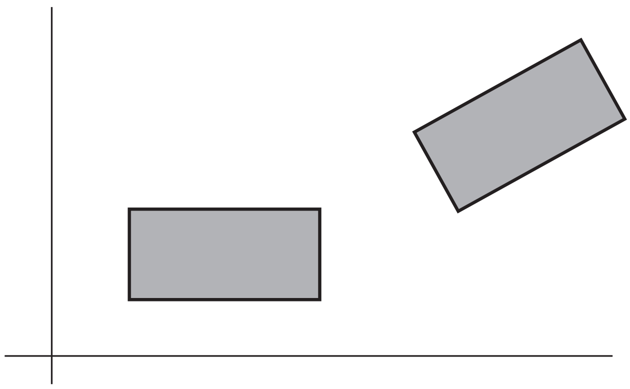
$$d_{\mathbf{0}}(x, y) = \begin{cases} 0 & \text{if } x = y; \\ d(x, \mathbf{0}) + d(\mathbf{0}, y) & \text{if } x \neq y. \end{cases}$$

Verify that d_0 is a metric on X. (Paris is at the origin $\mathbf{0}$.)

3. Let $X = \mathbb{R}^2$. For $x = (x_1, x_2)$ and $y = (y_1, y_2)$ define

$$d(x, y) = \begin{cases} 1/2 & \text{if } x_1 = y_1 \text{ and } x_2 \neq y_2 \text{ or if } x_1 \neq y_1 \text{ and } x_2 = y_2; \\ 1 & \text{if } x_1 \neq y_1 \text{ and } x_2 \neq y_2; \\ 0 & \text{otherwise.} \end{cases}$$

Verify that d is a metric and that the rectangles in the figure have different "area" if d is used to measure the length of sides.



4. Let (X, d) be a metric space. Consider the function $f : [0, \infty) \to [0, \infty)$ having the following properties:

(a) f is non-decreasing, i.e. $f(a) \leq f(b)$ if $0 \leq a < b$;

(b) $f(x) = 0$ if and only if $x = 0$;

(c) $f(a + b) \leq f(a) + f(b)$, $a, b \in [0, \infty)$.

If $x, y \in X$ define $d_f(x, y) = f(d(x, y))$. Show that d_f is a metric and that the functions $f(t) = kt$ where $k > 0$, $f(t) = t^\alpha$ where $0 < \alpha \leq 1$, and $f(t) = \frac{t}{1+t}$ for $t \geq 0$ have properties (a)–(c).

5. (**p-adic metric**) Let p be a prime number. Define the p-*adic absolute value function* $|\cdot|_p$ on $\mathbb{Q}$ by setting $|x|_p = 0$ when $x = 0$ and $|x|_p = p^{-k}$ when $x = p^k \cdot \frac{m}{n}$ where m, n are nonzero integers which are not divisible by p. Show that for $x, y \in \mathbb{Q}$,

$$|x + y|_p \leq \max\{|x|_p, |y|_p\}$$

and that $d(x, y) = |x - y|_p$ defines a metric on $\mathbb{Q}$. In fact, $d(x, z) \leq \max\{d(x, y), d(y, z)\}$. If d satisfies this condition which is stronger than the triangle inequality then d is called an *ultrametric.*

6. Let (X_i, d_i) be a metric space for $1 \leq i \leq n$ and let $X = \prod_{i=1}^n X_i$. Define

$$d(x, y) = \left[\sum_{i=1}^n d_i(x_i, y_i)^2\right]^{1/2},$$

$$\overline{d}(x, y) = \max\{d_i(x_i, y_i) \mid 1 \leq i \leq n\},$$

where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n) \in X$. Verify that d and $\overline{d}$ are metrics on X.

7. Fix a positive integer n. Denote by $\mathcal{P}_n$ the real vector space of all polynomials $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ with real coefficients a_i. For $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \in \mathcal{P}_n$ set

$$\|p\| = \max\{|a_0|, |a_1|, \ldots, |a_n|\}.$$

Verify that $\|\cdot\|$ is a norm on $\mathcal{P}_n$.

8. Let (X_n, d_n), $n \in \mathbb{N}$, be a sequence of metric spaces and let $X = \prod_{n\in\mathbb{N}} X_n$ be the cartesian product of the X_n's. (The elements of X are of the form $x = (x_1, x_2, \ldots)$ with $x_n \in X_n$.) For $x, y \in X$, define

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{d_n(x_n, y_n)}{1 + d_n(x_n, y_n)}.$$

Show that (X, d) is a metric space.

9. Sketch the open ball $B(0, 1)$ in the metric space $(\mathbb{R}^3, d_i)$, where d_i is defined by

$$d_1(x, y) = |x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3|$$

$$d_2(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}$$

$$d_\infty(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|, |x_3 - y_3|\}.$$

for $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3) \in \mathbb{R}^3$.

10. Set

$$d(n, m) = \left|\frac{1}{n} - \frac{1}{m}\right|$$

for $n, m \in \mathbb{N}$. Then d is a metric.

(a) Let $P \subset \mathbb{N}$ be the set of positive even numbers. Find $\text{diam}(P)$ and $\text{diam}(\mathbb{N} \setminus P)$ in $(\mathbb{N}, d)$.

(b) For a fixed $n \in \mathbb{N}$ find all elements of $B(2n, \frac{1}{2n})$ and $B(n, \frac{1}{2n})$.