

620-311 Metric Spaces: Problem Set 10

- Which of the following sets X are connected in \mathbb{R}^2 ?
 - Let $H = \{(x, y) \in \mathbb{R}^2 \mid xy = 1 \text{ and } x, y > 0\}$, $L = \{(x, 0) \mid x \in \mathbb{R}\}$, and $X = H \cup L$;
 - Let $C_n = \{(x, y) \in \mathbb{R}^2 \mid (x - 1/n)^2 + y^2 = 1/n^2\}$ for $n \in \mathbb{Z}$, and $X = \bigcup_{n \in \mathbb{N}} C_n$.
- Show that if A is a connected subspace of a topological space (X, \mathcal{T}) and if $A \subset B \subset \overline{A}$, then B is connected.
- If A and B are connected subsets of a topological space (X, \mathcal{T}) such that $\overline{A} \cap B \neq \emptyset$, then $A \cup B$ is connected.
- A point $p \in X$ is called a *cut point* if $X \setminus \{p\}$ is disconnected. Show that the property of having a cut point is a topological property. (A property of a topological space is a *topological property* if it is preserved under homeomorphisms.)
- Show that no two of the intervals (a, b) , $(a, b]$, and $[a, b]$ are homeomorphic.
- Show that \mathbb{R} and \mathbb{R}^2 are not homeomorphic (where \mathbb{R} and \mathbb{R}^2 are equipped with the usual topologies).
- Let $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ be the unit circle in \mathbb{R}^2 , and let $f : S^1 \rightarrow \mathbb{R}$ be a continuous function. Show that there exists $x \in S^1$ such that $f(x) = f(-x)$. [Hint: consider the function $g : S^1 \rightarrow \mathbb{R}$ where $g(x) = f(x) - f(-x)$.]
- Let A be a countable set. Show that $\mathbb{R}^2 \setminus A$ is path connected.
- Show that if A is an open connected subset of \mathbb{R}^n , then A is path connected. [Hint: Fix a point $x_0 \in A$ and consider the set U of all $x \in A$ which can be joined to x_0 by a path in A . Show that U and $A \setminus U$ are open.]
- A metric space (X, d_X) is called a **chain connected** if for every pair x, y of points in X and every $\varepsilon > 0$, there are finitely many points $x = x_0, x_1, x_2, \dots, x_n = y$ such that $d_X(x_{i+1}, x_i) < \varepsilon$ for $i = 0, 1, \dots, n - 1$. Prove that a compact, chain connected metric space is connected.