620-311 Metric Spaces: Problem Set 11

1. (a) Let X, Y be topological spaces and let $A \subset X$, $B \subset Y$ be closed subsets. Show that $A \times B$ is a closed subset of $X \times Y$ with the product topology.
(b) Give an example of a closed subset W of $\mathbb{R} \times \mathbb{R}$ such that the projection $\pi_1(W)$ is not closed in $\mathbb{R}$.

2. Let X be a topological space. Prove that X is Hausdorff if and only if the *diagonal* $\Delta = \{(x, x) : x \in X\}$ is a closed subset of $X \times X$ with the product topology.

3. Assume that X_j, $1 \leq j \leq k$ are non-empty topological spaces. Let $X = X_1 \times \cdots \times X_k$. Prove the following statements:
(a) If X is Hausdorff, then every X_j is Hausdorff;
(b) If X is normal, then every X_j is normal;
(c) If X is compact, then every X_j is compact;
(d) If X is connected, then every X_j is connected.

4. Which of the following subsets of $C(\mathbb{R}) = C(\mathbb{R}, \mathbb{R})$ are pointwise bounded? Which are equicontinuous?
(a) The set $\{f_n\}$, where $f_n(x) = x + \sin(nx)$;
(b) The set $\{g_n\}$, where $g_n(x) = n + \sin x$;
(c) The set $\{h_n\}$, where $h_n(x) = |x|^{1/n}$;
(d) The set $\{k_n\}$, where $k_n(x) = n \sin(x/n)$.

5. Let $\mathcal{F}$ be a set differentiable functions $f : [a, b] \to \mathbb{R}$ with the property that $|f'(x)| \leq M$ and $|f(x_0)| \leq M$ for some M and all $x \in [a, b]$ and some $x_0 \in [a, b]$. Show that the family $\overline{\mathcal{F}}$ is compact in the space $C[a, b]$.

6. Let X be a compact metric space and let $\{I_n\}$ be a sequence of isometries I_n from X onto X.
(a) Show that there exists a subsequence $\{I_{n_k}\}$ that converges to an isometry I.
(b) Do the inverse isometries $\{I_{n_k}^{-1}\}$ converge to I^{-1}?
(c) Use this to show that the group $O(3)$ of 3 by 3 orthogonal matrices is compact. (Recall that any 3 by 3 orthogonal matrix A defines an isometry $A : S^2 \to S^2$.)

7. (a) Give a proof of Urysohn's theorem in metric spaces.
(b) Give an elementary proof of Tietze's theorem in $\mathbb{R}$.
(c) Prove that a Hausdorff topological space X is normal if and only if Tietze's extension theorem holds for X.