620-311 Metric Spaces: Problem Set 2

1. Let $X = \mathbb{R}^2$. For $x = (x_1, x_2)$ and $y = (y_1, y_2) \in X$ define

$$d_M(x,y) = \begin{cases} |x_2 - y_2| & \text{if } x_1 = y_1; \\ |x_1 - y_1| + |x_2| + |y_2| & \text{if } x_1 \neq y_1. \end{cases}$$

Also define

$$d_K(x,y) = \begin{cases} \|x - y\| & \text{if } x = ty \text{ for some } t \in \mathbb{R}; \\ \|x\| + \|y\| & \text{otherwise.} \end{cases}$$

where $\|x\| = [\sum_{i=1}^{n} x_i^2]^{1/2}$.
(Can you give reasonable interpretations of the metrics d_M and d_K?)

Study the convergence of the sequence x_n in the spaces (X, d_M) and (X, d_K) if
(a) $x_n = (\frac{1}{n}, \frac{n}{n+1})$;
(b) $x_n = (\frac{n}{n+1}, \frac{n}{n+1})$;
(c) $x_n = (\frac{1}{n}, \sqrt{n+1} - \sqrt{n})$.

2. Let $\{x_n\}$ and $\{y_n\}$ be sequences in a metric space (X, d) such that $x_n \to x$ and $y_n \to y$ as $n \to \infty$. Prove that $d(x_n, y_n) \to d(x, y)$ as $n \to \infty$.

3. Metrics d and $\overline{d}$ defined on X are called *Lipschitz equivalent* if there exist positive constants m, M such that

$$m \cdot d(x,y) \leq \overline{d}(x,y) \leq M \cdot d(x,y)$$

for all $x, y \in X$.
(a) Show that if d and $\overline{d}$ are Lipschitz equivalent, then they are equivalent. Give an example of X and two equivalent metrics on X which are not Lipschitz equivalent.
(b) For $p \geq 1$ and $x, y \in \mathbb{R}^n$, the l^p metric is defined by

$$d_p(x,y) = \Big[\sum_{i=1}^{n} |x_i - y_i|^p\Big]^{1/p} = \|x - y\|_p.$$

Show that if $p, q \geq 1$, then d_p and d_q are Lipschitz equivalent. (Hint: compare these with $d_\infty(x,y) = \max(|x_1 - y_1|, \ldots, |x_n - y_n|)$.)

4. Consider the set $X = [-1, 1]$ as a metric subspace of $\mathbb{R}$ with the standard metric. Let
(a) $A = \{x \in X \mid 1/2 < |x| < 2\}$;
(b) $B = \{x \in X \mid 1/2 < |x| \leq 2\}$;
(c) $C = \{x \in \mathbb{R} \mid 1/2 \leq |x| < 1\}$;
(d) $D = \{x \in \mathbb{R} \mid 1/2 \leq |x| \leq 1\}$;
(e) $E = \{x \in \mathbb{R} \mid 0 < |x| \leq 1 \text{ and } 1/x \notin \mathbb{Z}\}$.
Classify the sets in (a)–(e) as open/closed in X and $\mathbb{R}$.

5. Consider $\mathbb{R}^2$ with the standard metric. Let

(a) $A = \{(x,y)|\ -1 < x \le 1 \text{ and } -1 < y < 1\}$;
(b) $B = \{(x,y)|\ xy = 0\}$;
(c) $C = \{(x,y)|\ x \in \mathbb{Q}, y \in \mathbb{R}\}$;
(d) $D = \{(x,y)| -1 < x < 1 \text{ and } y = 0\}$;
(e) $E = \bigcup_{n=1}^{\infty}\{(x,y)|x = 1/n \text{ and } |y| \le n\}$.

Sketch (if possible) and classify the sets in (a)–(e) as open/closed/neither in $\mathbb{R}^2$.

6. Find the interior, the closure and the boundary of each of the following subsets of $\mathbb{R}^2$ with the standard metric:

(a) $A = \{(x,y)) \mid x > 0 \text{ and } y \neq 0\}$;
(b) $B = \{(x,y) \mid x \in \mathbb{N}, y \in \mathbb{R}\}$;
(c) $C = A \cup B$;
(d) $D = \{(x,y) \mid x \text{ is rational}\}$;
(e) $F = \{(x,y) \mid x \neq 0 \text{ and } y \le 1/x\}$.

7. Let A be a subset of a metric space X. Is the interior of A equal to the interior of the closure of A? Is the closure of the interior of A equal to the closure of A itself?

8. Consider a collection $\{A_i\}_{i\in I}$ of subsets of a metric space X. Show that

$$\bigcup_{i\in I} A_i^\circ \subset \left(\bigcup_{i\in I} A_i\right)^\circ \qquad \overline{\bigcap_{i\in I} A_i} \subset \bigcap_{i\in I} \overline{A_i}$$

$$\left(\bigcap_{i\in I} A_i\right)^\circ \subset \bigcap_{i\in I} A_i^\circ \qquad \bigcup_{i\in I} \overline{A_i} \subset \overline{\bigcup_{i\in I} A_i}$$

9. Let (X,d) be a metric space. Show that if $A \subset X$, then

(a) $\overline{A} = A \cup \partial A$.
(b) $\partial A = \overline{A} \setminus A^\circ$ and $A^\circ = A \setminus \partial A$.
(c) A is closed if and only if $\partial A = A \setminus A^\circ$.
(d) A is open if and only if $\partial A = \overline{A} \setminus A$.

10. Let X and Y be metric spaces and A, B non-empty subsets of X and Y, respectively. Prove that

(a) If $A \times B$ is an open subset of $X \times Y$, then A and B are open in X and Y, respectively.
(b) If $A \times B$ is a closed subset of $X \times Y$, then A and B are closed in X and Y, respectively.