620-311 Metric Spaces: Problem Set 3

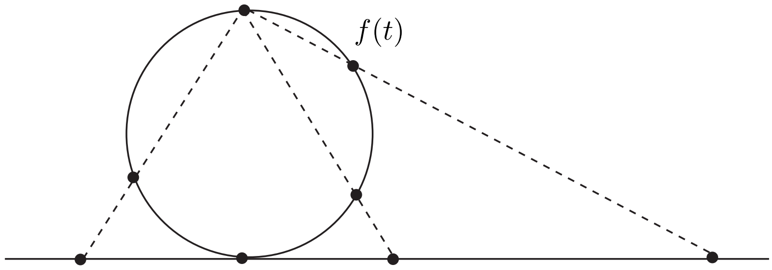
1. Let d and d' be equivalent metrics on X. Show that

(a) $A \subset X$ is closed in (X, d) if and only if A is closed in (X, d');
(b) $A \subset X$ is open in (X, d) if and only if A is open in (X, d').

2. Show that if $A \subset X$ then diam $A =$ diam $\overline{A}$. Does diam $A =$ diam A^0?

3. Let (X, d_X) and (Y, d_Y) be metric spaces and A, B are dense subsets of X and Y, respectively. Show that $A \times B$ is dense in $X \times Y$.

4 Let C be the circle in $\mathbb{R}^2$ with the centre at $(0, 1/2)$ and radius $1/2$. Let $X = C \setminus \{(0, 1)\}$. Define the function $f : \mathbb{R} \to X$ by defining $f(t)$ to be the point at which the line segment from $(t, 0)$ to $(0, 1)$ intersects X.



(a) Show that $f : \mathbb{R} \to X$ and $f^{-1} : X \to \mathbb{R}$ are continuous.
(b) Define for $s, t \in \mathbb{R}$

$$\rho(s, t) = \|f(s) - f(t)\|$$

where $\|\cdot\|$ is the standard norm in $\mathbb{R}^2$. Show that ρ defines a metric on $\mathbb{R}$ which is equivalent to the standard metric on $\mathbb{R}$.

5. Let $X = C[0, 1]$. Let $F : X \to \mathbb{R}$ be defined by $F(f) = f(0)$. Moreover, let $d_\infty(f, g) = \sup\{|f(x) - g(x)| \ : \ x \in [0, 1]\}$ and $d_1(f, g) = \int_0^1 |f(x) - g(x)| dx$.
Is F continuous when X is equipped with (a) the metric d_∞, (b) the metric d_1?

6. Let (X, d_X) and (Y, d_Y) be metric spaces. Show that $f : X \to Y$ is continuous if and only if

(a) $f(\overline{A}) \subset \overline{f(A)}$ for all subsets A of X, or
(b) $\overline{f^{-1}(B)} \subset f^{-1}(\overline{B})$ for all subsets B of Y.

7. Let (X, d) be a metric space and let a be a fixed point of X. Show that

$$|d(x, a) - d(y, a)| \leq d(x, y)$$

for all $x, y \in X$. Conclude that the function $f : X \to \mathbb{R}$ defined by $f(x) = d(x, a)$ is uniformly continuous.

8. Which of the following functions are uniformly continuous?

(a) $f(x) = \sin x$ on $[0, \infty)$
(b) $g(x) = \dfrac{1}{1 - x}$ on $(0, 1)$
(c) $h(x) = \sqrt{x}$ on $[0, \infty)$
(d) $k(x) = \sin(1/x)$, on $(0, 1)$

9. Which of the following sequences of functions converge uniformly on the interval $[0, 1]$?

(a) $f_n(x) = nx^2(1 - x)^n$
(b) $f_n(x) = n^2x(1 - x^2)^n$
(c) $f_n(x) = n^2x^3e^{-nx^2}$

10. Suppose that A is a dense subset of a metric space (X, d) and $f : A \to \mathbb{R}$ is uniformly continuous. Show that there exists exactly one continuous function $g : X \to \mathbb{R}$ satisfying $g(x) = f(x)$ for $x \in A$.
(Hint: You may need to use the completeness of $\mathbb{R}$.)