

620-311 Metric Spaces: Problem Set 4

1. Suppose that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in a metric space (X, d) . Prove that the sequence of real numbers $\{d(x_n, y_n)\}$ converges.
2. Suppose that $\{x_n\}$ is a sequence in a metric space (X, d) such that $d(x_n, x_{n+1}) \leq 2^{-n}$ for all $n \in \mathbb{N}$. Prove that $\{x_n\}$ is a Cauchy sequence.
3. Decide if the following metric spaces are complete:
 - (a) $((0, \infty), d)$, where $d(x, y) = |x^2 - y^2|$ for $x, y \in (0, \infty)$.
 - (b) $((-\pi/2, \pi/2), d)$, where $d(x, y) = |\tan x - \tan y|$ for $x, y \in (-\pi/2, \pi/2)$.
4. Let $X = (0, 1]$ be equipped with the usual metric $d(x, y) = |x - y|$. Show that (X, d) is not complete. Let $\tilde{d}(x, y) = \left| \frac{1}{x} - \frac{1}{y} \right|$ for $x, y \in X$. Show that \tilde{d} is a metric on X that is equivalent to d , and that (X, \tilde{d}) is complete.
5. Suppose that (X, d) and (Y, \tilde{d}) are metric spaces and that $f : X \rightarrow Y$ is a bijection such that both f and f^{-1} are uniformly continuous. Show that (X, d) is complete if and only if (Y, \tilde{d}) is complete.
6. [**Cantor's Intersection Theorem**] Let (X, d) be a metric space and let $\{F_n\}$ be a "decreasing" sequence of non-empty subsets of X satisfying $F_{n+1} \subseteq F_n$ for all n .
 - (a) Prove that if
 - (i) (X, d) is complete, (ii) each F_n is closed, (iii) $\text{diam}(F_n) \rightarrow 0$,then $\bigcap_{n \in \mathbb{N}} F_n$ consists of exactly one point.
 - (b) Show that, if any of (i)-(iii) is omitted, then $\bigcap_{n \in \mathbb{N}} F_n$ may be empty.
 - (c) Conversely, prove that if for every decreasing sequence $\{F_n\}$ of non-empty subsets satisfying (ii) and (iii), the intersection $\bigcap_{n \in \mathbb{N}} F_n$ is non-empty, then X is complete.
7. Let (X, d) be a complete metric space and let $f : X \rightarrow (0, \infty)$ be a continuous function. Prove that there exists a point x^* such that $f(y) \leq 2f(x^*)$ for all $y \in B(x^*, \frac{1}{\sqrt{f(x^*)}})$.

(Hint: Arguing by contradiction show that there exists a sequence $\{x_n\}$ with the following properties: $f(x_1) > 0$, $f(x_{n+1}) > 2f(x_n)$ for all $n \geq 1$ and $d(x_{n+1}, x_n) \leq \frac{1}{\sqrt{f(x_n)}}$. Then show that $\{x_n\}$ is Cauchy.)