Problem sheet 5

Problem 1
Which of the following maps are contractions?
(a) $f : \mathbb{R} \rightarrow \mathbb{R}$, \( f(x) = e^{-x} \);
(b) $f : [0, \infty) \rightarrow [0, \infty)$, \( f(x) = e^{-x} \);
(c) $f : [0, \infty) \rightarrow [0, \infty)$, \( f(x) = e^{-e^x} \);
(d) $f : \mathbb{R} \rightarrow \mathbb{R}$, \( f(x) = \cos x \);
(e) $f : \mathbb{R} \rightarrow \mathbb{R}$, \( f(x) = \cos(\cos x) \).

Problem 2
Consider the map $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ given by
\[
 f(x, y) = \frac{1}{10}(8x + 8y, x + y), \quad (x, y) \in \mathbb{R}^2.
\]
Recall metrics $d_1((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$, $d_2((x_1, y_1), (x_2, y_2)) = [(x_1 - x_2)^2 + (y_1 - y_2)^2]^{1/2}$ and $d_\infty((x_1, y_1), (x_2, y_2)) = \max\{|x_1 - x_2|, |y_1 - y_2|\}$. Is $f$ a contraction with respect to $d_1$? $d_2$? $d_\infty$?

Problem 3
(a) Consider $X = (0, a]$ with the usual metric and $f(x) = x^2$ for $x \in X$. Find values of $a$ for which $f$ is a contraction and show that $f : X \rightarrow X$ does not have a fixed point.

(b) Consider $X = [1, \infty)$ with the usual metric and let $f(x) = x + \frac{1}{x}$ for $x \in X$. Show that $f : X \rightarrow X$ and $d(f(x), f(y)) < d(x, y)$ for all $x \neq y$, but $f$ does not have a fixed point. Reconcile (a) and (b) with Banach fixed point theorem.

Problem 4
Let $(X, d)$ be a complete metric space and $f : X \rightarrow X$ be a function such that
\[
d(f(x), f(y)) \leq \alpha d(x, y)
\]
for all $x, y \in \overline{B}(x_0, r_0)$, where $0 < \alpha < 1$ and $d(x_0, f(x_0)) \leq (1 - \alpha) \cdot r_0$. Prove that $f$ has a unique fixed point $p \in \overline{B}(x_0, r_0)$.

Problem 5
(a) Show that there is exactly one continuous function $f : [0, 1] \rightarrow \mathbb{R}$ which satisfies the equation
\[
 [f(x)]^3 - e^x[f(x)]^2 + \frac{f(x)}{2} = e^x.
\]
(Hint: rewrite the equation as $f(x) = e^x + \frac{1}{2} \frac{f(x)}{1 + f(x)^2}$.)
(b) Consider $C[0, a]$ with $a < 1$ and $T : C[0, a] \to C[0, a]$ given by

$$(Tf)(t) = \sin t + \int_0^t f(s)ds, \quad t \in [0, a].$$

Show that $T$ is a contraction. What is the fixed point of $T$?

(c) Find all $f \in C[0, \pi]$ which satisfy the equation

$$3f(t) = \int_0^t \sin(t - s)f(s)ds.$$ 

(d) Let $g \in C[0, 1]$. Show that there exists exactly one $f \in C[0, 1]$ which solves the equation

$$f(x) + \int_0^1 e^{x-y-1}f(y)dy = g(x), \quad \text{forall } x \in [0, 1].$$

(Hint: Consider the metric $d(f, h) = \sup\{|e^{-x}f(x) - h(x)| : x \in [0, 1]\}.$)

Problem 6

Let $\{f_k\}$ be a sequence of linear maps $f_k : \mathbb{R}^n \to \mathbb{R}^m$ which are not identically zero, that is, for every $k \in \mathbb{N}$ there is $x = x_k$ such that $f_k(x) \neq 0$. Show that there is $x$ (not depending on $k$) such that $f_k(x) \neq 0$ for all $k \in \mathbb{N}$.

Problem 7

Let $\{f_n\}$ be a sequence of continuous functions $f_n : \mathbb{R} \to \mathbb{R}$ having the property that $\{f_n(x)\}$ is unbounded for all $x \in \mathbb{Q}$. Prove that there is at least one $x \in \mathbb{Q}^c$ such that $\{f_n(x)\}$ is unbounded.

Problem 8

Let $(X, d)$ be a complete metric space and let $(Y, \tilde{d})$ be a metric space. Let $\{f_n\}$ be a sequence of continuous functions from $X$ to $Y$ such that $\{f_n(x)\}$ converges for every $x \in X$. Prove that for every $\varepsilon > 0$ there exist $k \in \mathbb{N}$ and a non-empty open subset $U$ of $X$ such that $\tilde{d}(f_n(x), f_m(x)) < \varepsilon$ for all $x \in U$ and all $n, m \geq k$.

Problem 9

Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function having the property that for every $x$ there is a non-negative integer $n = n_x$ such that $f^{(n_x)}(x) = 0$. Prove that $f$ is a polynomial on $\mathbb{R}$.

Suggestion:

(a) First show that any open subset of $\mathbb{R}$ is the union of pairwise disjoint open intervals. (In fact, at most countably many of them.)

(b) Show that if $f$ is smooth and is a polynomial on $(a, b)$ and a polynomial on $(b, c)$, then $f$ is a polynomial on $(a, c)$.

Let $O$ be a collection of all points $x \in \mathbb{R}$ for which there exists an open interval $I_x$
containing $x$ such that $f$ is a polynomial on $I_x$. Let $F = \mathbb{R} \setminus O$ and $F_n = \{x \in F| f^{(n)}(x) = 0\}$.

(c) Show that $O$ is open. Hence, by (a), $O = \bigcup_i I_i$, where the $I_i$ are open intervals. Show that for every interval $I_i$ there is a polynomial $P$ such that $f = P$ on $I_i$. To see this you may use the following reasoning. Let $I_i = (a, b)$ and $x_0 \in (a, b)$. Set $I = \{x \in (x_0, b) : f = P|_{[x_0, x]}\}$ and let $c = sup I$. Arguing by contradiction and using (b), show that $c = b$. Note that this implies the result if we can show that $F = \emptyset$.

(d) Show that $F$ is closed and does not have isolated points (use (c)).

(e) Assume that $F \neq \emptyset$ and use Baire’s theorem to conclude that there is $N \in \mathbb{N}$ and an open interval $I$ in $\mathbb{R}$ such that $I \cap F \subset F_N$.

(f) Use (e) and (d) to show that $I \cap F \subset F_n$ for all $n \geq N$.

(g) Show that $f^{(N)} = 0$ on $I \cap I_i$ provided that $I \cap I_i \neq \emptyset$ and conclude that $F = \emptyset$. To see this take note that $I \cap I_i = (a, b)$. Show that $a \in F$ and use Taylor’s formula.

If you would like to see more applications of Baire’s theorem, you may look at the book by R. Boas, A primer of real functions.