Problem sheet 5

Problem 1
Which of the following maps are contractions?

(a) $f : \mathcal{R} \to \mathcal{R}$, $f(x) = e^{-x}$;

(b) $f : [0, \infty) \to [0, \infty)$, $f(x) = e^{-x}$;

(c) $f : [0, \infty) \to [0, \infty)$, $f(x) = e^{-e^x}$;

(d) $f : \mathcal{R} \to \mathcal{R}$, $f(x) = \cos x$;

(e) $f : \mathcal{R} \to \mathcal{R}$, $f(x) = \cos(\cos x)$.

Problem 2
Consider the map $f : \mathcal{R}^2 \to \mathcal{R}^2$ given by

$$f(x, y) = \frac{1}{10}(8x + 8y, x + y),\ (x, y) \in \mathcal{R}^2.$$

Recall metrics $d_1((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$, $d_2((x_1, y_1), (x_2, y_2)) = [|x_1 - x_2|^2 + |y_1 - y_2|^2]^{1/2}$ and $d_\infty((x_1, y_1), (x_2, y_2)) = \max\{|x_1 - x_2|, |y_1 - y_2|\}$. Is f a contraction with respect to d_1? d_2? d_∞?

Problem 3

(a) Consider $X = (0, a]$ with the usual metric and $f(x) = x^2$ for $x \in X$. Find values of a for which f is a contraction and show that $f : X \to X$ does not have a fixed point.

(b) Consider $X = [1, \infty)$ with the usual metric and let $f(x) = x + \frac{1}{x}$ for $x \in X$. Show that $f : X \to X$ and $d(f(x), f(y)) < d(x, y)$ for all $x \neq y$, but f does not have a fixed point. Reconcile (a) and (b) with Banach fixed point theorem.

Problem 4
Let (X, d) be a complete metric space and $f : X \to X$ be a function such that

$$d(f(x), f(y)) \leq \alpha d(x, y)$$

for all $x, y \in \overline{B}(x_0, r_0)$, where $0 < \alpha < 1$ and $d(x_0, f(x_0)) \leq (1 - \alpha) \cdot r_0$. Prove that f has a unique fixed point $p \in \overline{B}(x_0, r_0)$.

Problem 5

(a) Show that there is exactly one continuous function $f : [0, 1] \to \mathcal{R}$ which satisfies the equation

$$[f(x)]^3 - e^x [f(x)]^2 + \frac{f(x)}{2} = e^x.$$

(Hint: rewrite the equation as $f(x) = e^x + \frac{1}{2} \frac{f(x)}{1 + f(x)^2}$.)

(b) Consider $C[0,a]$ with $a < 1$ and $T : C[0,a] \to C[0,a]$ given by

$$(Tf)(t) = \sin t + \int_0^t f(s)ds, \ t \in [0,a].$$

Show that T is a contraction. What is the fixed point of T?

(c) Find all $f \in C[0,\pi]$ which satisfy the equation

$$3f(t) = \int_0^t \sin(t-s)f(s)ds.$$

(d) Let $g \in C[0,1]$. Show that there exists exactly one $f \in C[0,1]$ which solves the equation

$$f(x) + \int_0^1 e^{x-y-1} f(y)dy = g(x), forall x \in [0,1].$$

(Hint: Consider the metric $d(f,h) = \sup\{e^{-x}|f(x) - h(x)| : x \in [0,1]\}$.)

Problem 6

Let $\{f_k\}$ be a sequence of linear maps $f_k : \mathcal{R}^n \to \mathcal{R}^m$ which are not identically zero, that is, for every $k \in \mathcal{N}$ there is $x = x_k$ such that $f_k(x) \neq 0$. Show that there is x (not depending on k) such that $f_k(x) \neq 0$ for all $k \in \mathcal{N}$.

Problem 7

Let $\{f_n\}$ be a sequence of continuous functions $f_n : \mathcal{R} \to \mathcal{R}$ having the property that $\{f_n(x)\}$ is unbounded for all $x \in \mathcal{Q}$. Prove that there is at least one $x \in \mathcal{Q}^c$ such that $\{f_n(x)\}$ is unbounded.

Problem 8

Let (X,d) be a complete metric space and let $(Y,\tilde{d})$ be a metric space. Let $\{f_n\}$ be a sequence of continuous functions from X to Y such that $\{f_n(x)\}$ converges for every $x \in X$. Prove that for every $\varepsilon > 0$ there exist $k \in \mathcal{N}$ and a non-empty open subset U of X such that $\tilde{d}(f_n(x), f_m(x)) < \varepsilon$ for all $x \in U$ and all $n, m \geq k$.

Problem 9

Let $f : \mathcal{R} \to \mathcal{R}$ be a smooth function having the property that for every x there is a non-negative integer $n = n_x$ such that $f^{(n_x)}(x) = 0$. Prove that f is a polynomial on $\mathcal{R}$.
Suggestion:

(a) First show that any open subset of $\mathcal{R}$ is the union of pairwise disjoint open intervals. (In fact, at most countably many of them.)

(b) Show that if f is smooth and is a polynomial on (a,b) and a polynomial on (b,c), then f is a polynomial on (a,c).

Let O be a collection of all points $x \in \mathcal{R}$ for which there exists an open interval I_x

containing x such that f is a polynomial on I_x. Let $F = \mathcal{R} \setminus O$ and $F_n = \{x \in F |\ f^{(n)}(x) = 0\}$.

(c) Show that O is open. Hence, by (a), $O = \bigcup_i I_i$, where the I_i are open intervals. Show that for every interval I_i there is a polynomial P such that $f = P$ on I_i. To see this you may use the following reasoning. Let $I_i = (a, b)$ and $x_0 \in (a, b)$. Set $I = \{x \in (x_0, b) : f = P|_{[x_0, x]}\}$ and let $c = supI$. Arguing by contradiction and using (b), show that $c = b$. Note that this implies the result if we can show that $F = \emptyset$.

(d) Show that F is closed and does not have isolated points (use (c)).

(e) Assume that $F \neq \emptyset$ and use Baire's theorem to conclude that there is $N \in \mathcal{N}$ and an open interval I in $\mathcal{R}$ such that $I \cap F \subset F_N$.

(f) Use (e) and (d) to show that $I \cap F \subset F_n$ for all $n \geq N$.

(g) Show that $f^{(N)} = 0$ on $I \cap I_i$ provided that $I \cap I_i \neq \emptyset$ and conclude that $F = \emptyset$. To see this take note that $I \cap I_i = (a, b)$. Show that $a \in F$ and use Taylor's formula.

If you would like to see more applications of Baire's theorem, you may look at the book by R. Boas, A primer of real functions*.*