1. On $\mathbb{R}$ consider the metrics:
   
   \[ d_1(x, y) = |\arctan x - \arctan y|, \]
   \[ d_2(x, y) = |x^3 - y^3|. \]

   With which of these metrics is $\mathbb{R}$ complete? If $(\mathbb{R}, d_i)$ is not complete find its completion.

2. Which of the following subsets of $\mathbb{R}$ and $\mathbb{R}^2$ are compact? ($\mathbb{R}$ and $\mathbb{R}^2$ are considered with the usual metrics).
   
   (a) $A = \mathbb{Q} \cap [0, 1]$
   
   (b) $B = \{ (x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1 \}$
   
   (c) $C = \{ (x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1 \}$
   
   (d) $D = \{ (x, y) : |x| + |y| \leq 1 \}$
   
   (e) $E = \{ (x, y) : x \geq 1 \text{ and } 0 \leq y \leq 1/x \}$

3. Prove that if $A_1, \ldots, A_k$ are compact subsets of a metric space $(X, d)$, then $\bigcup_{i=1}^k A_i$ is compact.

4. Prove that if $A_i$ is a compact subset of the metric space $(X_i, d_i)$ for $i = 1, \ldots, k$, then $A_1 \times \cdots \times A_k$ is a compact subset of $X = X_1 \times \cdots X_k$ with the product metric $d$.

5. Let $A$ be a non-empty compact subset of a metric space $(X, d)$. Prove:

   (a) If $x \in X$, then there exists $a \in A$ such that $d(x, a) = d(x, A)$;
   
   (b) If $A \subseteq U$ and $U$ is open, then there is $\varepsilon > 0$ such that $\{ x \in X : d(x, A) < \varepsilon \} \subseteq U$.
   
   (c) If $B$ is closed and $A \cap B = \emptyset$, then $d(A, B) > 0$.

   *Hint:* Recall that $(x, y) \mapsto d(x, y)$ is continuous from $X \times X \to [0, \infty)$.

6. Let $f : X \to \mathbb{R}$. Call a function $f$ upper semicontinuous, abbreviated u.s.c., if for every $r \in \mathbb{R}$, $\{ x \in X : f(x) < r \}$ is open. Similarly, $f$ is lower semicontinuous, abbreviated l.s.c., if for every $r \in \mathbb{R}$, $\{ x \in X : f(x) > r \}$ is open. Assume that $X$ is compact. Show that every u.s.c. function assumes a maximum value and every l.s.c. function assumes a minimum value.

7. Call a map $f : X \to X$ weak contraction if $d(f(x), f(y)) < d(x, y)$ for all $x \neq y$. Prove that if $X$ is compact and $f$ is a weak contraction, then $f$ has a unique fixed point.
The next problem gives a different construction of the completion of a metric space \((X, d)\).

An **equivalence relation** on a set \(X\) is a relation \(\sim\) having the following three properties:

(a) **(Reflexivity)** \(x \sim x\) for every \(x \in X\).

(b) **(Symmetry)** If \(x \sim y\), then \(y \sim x\).

(c) **(Transitivity)** If \(x \sim y\) and \(y \sim z\), then \(x \sim z\).

The **equivalence class** determined by \(x\), and denoted by \([x]\), is defined by \([x] = \{y \in X : y \sim x\}\). We have \([x] = [y]\) if and only if \(x \sim y\), and \(X\) is a disjoint union of these equivalence classes.

8. Let \((X, d)\) be a metric space and let \(X^\ast\) be the set of Cauchy sequences \(x = \{x_n\}\) in \((X, d)\). Define a relation \(\sim\) in \(X^\ast\) by declaring \(x = \{x_n\} \sim y = \{y_n\}\) to mean \(d(x_n, y_n) \to 0\).

(a) Show that \(\sim\) is an equivalence relation.

Denote by \([x]\) the equivalence class of \(x \in X^\ast\), and let \(\widetilde{X}\) denote the set of these equivalence classes.

(b) Show that if \(x = \{x_n\}\) and \(y = \{y_n\} \in X^\ast\), then \(\lim_{n \to \infty} d(x_n, y_n)\) exists.

Show that if \(x' = \{x'_n\} \in [x]\) and \(y' = \{y'_n\} \in [y]\), then

\[
\lim_{n \to \infty} d(x_n, y_n) = \lim_{n \to \infty} d(x'_n, y'_n).
\]

For \([x], [y] \in \widetilde{X}\), define

\[
D([x], [y]) = \lim_{n \to \infty} d(x_n, y_n).
\]

Note that the definition of \(D\) is unambiguous in view of the above equality.

(c) Show that \((\widetilde{X}, D)\) is a complete metric space.

**Hint:** Let \([x^n]\) be Cauchy in \((\widetilde{X}, D)\). Then \(x^n = \{x^n_1, x^n_2, x^n_3, \ldots\}\) is Cauchy in \((X, d)\). So for every \(n \in \mathbb{N}\), there exists \(k_n \in \mathbb{N}\) such that

\[
d(x^n_m, x^n_{k_n}) < 1/n \quad \text{for all } m \geq k_n.
\]

Set \(x = \{x^1_{k_1}, x^2_{k_2}, x^3_{k_3}, \ldots\}\). Then show that \(x\) is Cauchy in \((X, d)\) and \(D([x^n], [x]) \to 0\).

(d) If \(x \in X\), let \(\varphi(x)\) be the equivalence class of the constant sequence \(x = (x, x, x, \ldots)\). That is, \(\varphi(x) = [x] = \{x, x, x, \ldots\}\). Show that \(\varphi : X \to \varphi(X)\) is an isometry.

(e) Show that \(\varphi(X)\) is dense in \((\widetilde{X}, D)\).

**Hint:** Let \([x] \in \widetilde{X}\) with \(x = \{x_1, x_2, x_3, \ldots\}\). Denote by \(x^n\) the constant sequence \(\{x_n, x_n, x_n, \ldots\}\) and show that \(D([x^n], [x]) \to 0\).