620-311 Metric Spaces: Problem Set 6

1. On $\mathbb{R}$ consider the metrics:

$$d_1(x,y) = |\arctan x - \arctan y|,$$
$$d_2(x,y) = \left|x^3 - y^3\right|.$$

With which of these metrics is $\mathbb{R}$ complete? If $(\mathbb{R}, d_i)$ is not complete find its completion.

2. Which of the following subsets of $\mathbb{R}$ and $\mathbb{R}^2$ are compact? ($\mathbb{R}$ and $\mathbb{R}^2$ are considered with the usual metrics).
(a) $A = \mathbb{Q} \cap [0,1]$
(b) $B = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$
(c) $C = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}\}$
(d) $D = \{(x,y) : |x| + |y| \leq 1\}$
(e) $E = \{(x,y) : x \geq 1 \text{ and } 0 \leq y \leq 1/x\}$

3. Prove that if $A_1, \ldots, A_k$ are compact subsets of a metric space (X,d), then $\bigcup_{i=1}^k A_i$ is compact.

4. Prove that if A_i is a compact subset of the metric space (X_i, d_i) for $i = 1, \ldots, k$, then $A_1 \times \cdots \times A_k$ is a compact subset of $X = X_1 \times \cdots X_k$ with the product metric d.

5. Let A be a non-empty compact subset of a metric space (X,d). Prove:

(a) If $x \in X$, then there exists $a \in A$ such that $d(x,a) = d(x,A)$;
(b) If $A \subset U$ and U is open, then there is $\varepsilon > 0$ such that $\{x \in X : d(x,A) < \varepsilon\} \subset U$.
(c) If B is closed and $A \cap B = \emptyset$, then $d(A,B) > 0$.

Hint: Recall that $(x,y) \mapsto d(x,y)$ is continuous from $X \times X \to [0,\infty)$.

6. Let $f : X \to \mathbb{R}$. Call a function f **upper semicontinuous**, abbreviated u.s.c., if for every $r \in \mathbb{R}$, $\{x \in X \mid f(x) < r\}$ is open. Similarly, f is **lower semicontinuous**, abbreviated l.s.c., if for every $r \in \mathbb{R}$, $\{x \in X \mid f(x) > r\}$ is open. Assume that X is compact. Show that every u.s.c. function assumes a maximum value and every l.s.c. function assumes a minimum value.

7. Call a map $f : X \to X$ **weak contraction** if $d(f(x), f(y)) < d(x,y)$ for all $x \neq y$. Prove that if X is compact and f is a weak contraction, then f has a unique fixed point.

The next problem gives a different construction of the completion of a metric space (X, d).

An **equivalence relation** on a set X is a relation $\sim$ having the following three properties:

(a) (Reflexivity) $x \sim x$ for every $x \in X$.
(b) (Symmetry) If $x \sim y$, then $y \sim x$.
(c) (Transitivity) If $x \sim y$ and $y \sim z$, then $x \sim z$.

The **equivalence class** determined by x, and denoted by $[x]$, is defined by $[x] = \{y \in X : y \sim x\}$. We have $[x] = [y]$ if and only if $x \sim y$, and X is a disjoint union of these equivalence classes.

8. Let (X, d) be a metric space and let X^* be the set of Cauchy sequences $\mathbf{x} = \{x_n\}$ in (X, d). Define a relation $\sim$ in X^* by declaring $\mathbf{x} = \{x_n\} \sim \mathbf{y} = \{y_n\}$ to mean $d(x_n, y_n) \to 0$.

(a) Show that $\sim$ is an equivalence relation.

Denote by $[\mathbf{x}]$ the equivalence class of $\mathbf{x} \in X^*$, and let $\widetilde{X}$ denote the set of these equivalence classes.

(b) Show that if $\mathbf{x} = \{x_n\}$ and $\mathbf{y} = \{y_n\} \in X^*$, then $\lim_{n\to\infty} d(x_n, y_n)$ exists. Show that if $\mathbf{x}' = \{x'_n\} \in [\mathbf{x}]$ and $\mathbf{y}' = \{y'_n\} \in [\mathbf{y}]$, then

$$\lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(x'_n, y'_n).$$

For $[\mathbf{x}], [\mathbf{y}] \in \widetilde{X}$, define

$$D([\mathbf{x}], [\mathbf{y}]) = \lim_{n\to\infty} d(x_n, y_n).$$

Note that the definition of D is unambiguous in view of the above equality.

(c) Show that $(\widetilde{X}, D)$ is a complete metric space.

Hint: Let $[\mathbf{x}^n]$ be Cauchy in $(\widetilde{X}, D)$. Then $\mathbf{x}^n = \{x_1^n, x_2^n, x_3^n, \dots\}$ is Cauchy in (X, d). So for every $n \in \mathbb{N}$, there exists $k_n \in \mathbb{N}$ such that

$$d(x_m^n, x_{k_n}^n) < 1/n \quad \text{for all } m \geq k_n.$$

Set $\mathbf{x} = \{x_{k_1}^1, x_{k_2}^2, x_{k_3}^3, \dots\}$. Then show that $\mathbf{x}$ is Cauchy in (X, d) and $D([\mathbf{x}^n], [\mathbf{x}]) \to 0$.

(d) If $x \in X$, let $\varphi(x)$ be the equivalence class of the constant sequence $\mathbf{x} = (x, x, x, \dots)$. That is, $\varphi(x) = [\mathbf{x}] = [\{x, x, x, \dots\}]$. Show that $\varphi : X \to \varphi(X)$ is an isometry.

(e) Show that $\varphi(X)$ is dense in $(\widetilde{X}, D)$.

Hint: Let $[\mathbf{x}] \in \widetilde{X}$ with $\mathbf{x} = \{x_1, x_2, x_3, \dots\}$. Denote by $\mathbf{x}^n$ the constant sequence $\{x_n, x_n, x_n, \dots\}$ and show that $D([\mathbf{x}^n], [\mathbf{x}]) \to 0$.