620-311 Metric Spaces: Problem Set 7

1. Consider the following spaces:

(a) $\mathbb{R}$ with the metric $d_1(x,y) = \dfrac{|x-y|}{1+|x-y|}$;

(b) $\mathbb{R}$ with the metric $d_2(x,y) = |\arctan x - \arctan y|$;

(c) $\mathbb{R}$ with the metric $d_3(x,y) = 0$ if $x = y$ and $d(x,y) = 1$ if $x \neq y$.

Is $(\mathbb{R}, d_i)$ compact?

2. Use the Heine-Borel property to prove that if $f : X \to Y$ is a continuous mapping between metric spaces and X is compact then f is uniformly continuous.

3. A family $\{F_i\}_{i\in I}$ is said to have the **finite intersection property** if for every finite subset J of I, $\bigcap_{i\in J} F_i \neq \emptyset$. Show that X is compact if and only if for every family $\{F_i\}_{i\in I}$ of closed subsets of X having the finite intersection property, the intersection $\bigcap_{i\in I} F_i \neq \emptyset$.

4. Consider $C[0,1]$ with the usual d_∞ metric. Let

$$A = \{f \in C[0,1] : 0 = f(0) \leq f(t) \leq f(1) = 1 \text{ for all } t \in [0,1]\}.$$

Show that there is no finite 1/2-net for A.

5. Show that if $A \subset X$ is totally bounded, then $\overline{A}$ is also totally bounded.

6. Show that a metric space (X,d) is totally bounded if and only if every sequence $\{x_n\} \subseteq X$ contains a Cauchy subsequence.

7. Let X be a totally bounded metric space and Y a metric space. Assume that $f : X \to Y$ is a bijection. Show that if f and f^{-1} are uniformly continuous, then Y is totally bounded.

8. [**Lebesgue number lemma**] Let (X,d) be a compact metric space and let $\{U_i\}_{i\in I}$ be an open covering of X. Prove that there exists $\delta > 0$ such that for every subset $A \subset X$ with $\mathrm{diam}(A) < \delta$ there exists $i \in I$ such that $A \subset U_i$.
(δ is called a "Lebesgue number" for the covering.)

9. Let (X,d) be a compact metric space. Assume that $f : X \to X$ preserves distance, that is,

$$d(f(x), f(y)) = d(x,y)$$

for every $x, y \in X$. Show that f is a bijection.

Hint: Assume that $f(X) \neq X$. So there exists $a \in X \setminus f(X)$. Since f is continuous and X is compact, $f(X)$ is compact. So $d(a, f(X)) = r > 0$. Consider a sequence $x_n = f^n(a)$.