620-311 Metric Spaces: Problem Set 8

1. (a) List all possible topologies on the set with two elements $X = \{a, b\}$.
(b) Find the closure and the interior of the sets $\{a\}$ and $\{b\}$ in each topology.
(c) Which of these topological spaces are homeomorphic?
(d) Which of these topological spaces are Hausdorff?

2. Show that each of the following defines a topological space $(X, \mathcal{T})$.
(a) Let X be an infinite set and define $\mathcal{T}$ by

$$\mathcal{T} = \{A \subseteq X : A = \emptyset \text{ or } A = X \text{ or } X \setminus A \text{ is countable}\}.$$

($\mathcal{T}$ is called *co-countable topology* or *countable complement* topology)
(b) Let $X = \mathbb{R}$ and let

$$\mathcal{T} = \{A \subseteq \mathbb{R} : A = \emptyset \text{ or } A = \mathbb{R} \text{ or } A = (a, \infty) \text{ with } a \in \mathbb{R}\}.$$

3. Let $\mathcal{B} = \{[a, b)|\ a, b \in \mathbb{R}\}$. Show that $\mathcal{B}$ is a basis for a topology on $\mathbb{R}$. This topology, denoted $\mathcal{T}_l$, is called the **lower-limit topology** on $\mathbb{R}$. Show that the lower-limit topology is larger than the usual topology on $\mathbb{R}$. Decide whether the sets $[a, b)$, (a, b), $(a, b]$ and $[a, b]$ are closed in $(\mathbb{R}, \mathcal{T}_l)$. Hence find the closure of each set.

4. Let $\mathcal{T} = \{A \subseteq \mathbb{R}|\ 0 \notin A \text{ or } A = \mathbb{R}\}$. Show that $\mathcal{T}$ is a topology on $\mathbb{R}$. What are the closed sets in $(\mathbb{R}, \mathcal{T})$? What is $\overline{\{1\}}$? Is this topology Hausdorff?

5. Let A, B be a subsets of a topological space $(X, \mathcal{T})$. Show that $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$ and $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

6. Let $A \subset X$ where $(X, \mathcal{T})$ is a topological space. Show that $\overline{X \setminus A} = X \setminus A^\circ$ and $(X \setminus A)^\circ = X \setminus \overline{A}$.

7. Let X be infinite set and let $\mathcal{T}$ be a co-finite topology on X. Show that any continuous function $f : X \to \mathbb{R}$ is constant. ($\mathbb{R}$ is equipped with the usual metric topology).

8. Let X and Y be topological spaces and let $\mathcal{B}$ be a basis of open sets for Y. Show that a function $f : X \to Y$ is continuous if and only if $f^{-1}(U)$ is open in X for every $U \in \mathcal{B}$.

9. Show that compositions of homeomorphisms are homeomorphisms.

10. (a) Show that (a, b) is homeomorphic to (c, d), (c, ∞) and $\mathbb{R}$. (All spaces are equipped with the usual topology).
(b) Show that $\mathbb{R}^2 \setminus \{(0, 0)\}$ is homeomorphic to $\mathbb{R}^2 \setminus \overline{B}((0, 0), 1)$ with the usual topology.