620-311 Metric Spaces: Problem Set 9

1. Call a space X *discrete* if it is equipped with the discrete topology and *trivial* if it is equipped with the the trivial topology. Now prove:
(a) If X is discrete, then every function $f : X \to Y$, where Y is any topological spaces, is continuous.
(b) If X is trivial with at least two elements, then there exists a topological space Y and a function $f : X \to Y$ that is not continuous.
(c) If Y is trivial, then every function $f : X \to Y$, where X is any topological space, is continuous.
(d) If Y is discrete and contains at least two elements, then there exists a topological space X and a function $f : X \to Y$ that is not continuous.

2. Show that the unit open ball $B = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$ in $\mathbb{R}^2$ is homeomorphic to the open square $C = (-1, 1) \times (-1, 1)$ using the usual topology.

3. Show that the unit open ball $B = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$ in $\mathbb{R}^2$ is homeomorphic to the open right half plane $H = (0, \infty) \times \mathbb{R}$ using the usual topology.

4. Let $f, g : X \to Y$ be continuous functions between topological spaces, and assume that Y is Hausdorff. Prove that $\{x \in X : f(x) = g(x)\}$ is a closed subset of X.

5. Let $(X, \mathcal{T})$ be a compact topological space and let A, B are closed subsets of $(X, \mathcal{T})$. Show that $A \cup B$ is compact.

6. Let $X = (0, 1)$ and let

$$\mathcal{T} = \{A \subseteq \mathbb{R} \mid A = \emptyset \text{ or } A = (0, 1) \text{ or } A = (0, 1 - 1/n) \text{ for } n \geq 2\}.$$

Show that every open set A other than X is compact. Is X compact?

7. Let $\mathcal{T}$ be the co-countable topology on $\mathbb{R}$, that is,

$$\mathcal{T} = \{A \subseteq \mathbb{R} \mid A = \emptyset \text{ or } \mathbb{R} \setminus A \text{ is countable}\}.$$

Is $[0, 1]$ compact in $(\mathbb{R}, \mathcal{T})$? What are the compact sets in $(\mathbb{R}, \mathcal{T})$?

8. Suppose that $\{X_i\}_{i \in I}$ is a family of non-empty closed sets of a Hausdorff topological space X, and that at least one of the X_i is compact. In addition, assume that the family has the property that for any two $i, j \in I$ either $X_i \subset X_j$ or $X_j \subset X_i$. Show that $\bigcap_{i \in I} X_i \neq \emptyset$.

9. Let (X, d) be a metric space and let $\mathcal{H}$ be the collection of all non-empty compact subsets of X. Define

$$D(A, B) = \max\{\sup\{d(x, A) \mid x \in B\}, \sup\{d(x, B) \mid x \in A\}\}.$$

(a) Show that D defines a metric on $\mathcal{H}$. The metric D is called the **Hausdorff metric**.
(b) Show that if (X, d) is complete, then $(\mathcal{H}, D)$ is complete.
(b) Show that if (X, d) is totally bounded, then $(\mathcal{H}, D)$ is totally bounded.
(b) Show that if (X, d) is compact, then $(\mathcal{H}, D)$ is compact.