1. Call a space \( X \) *discrete* if it is equipped with the discrete topology and *trivial* if it is equipped with the trivial topology. Now prove:

(a) If \( X \) is discrete, then every function \( f : X \to Y \), where \( Y \) is any topological space, is continuous.

(b) If \( X \) is trivial with at least two elements, then there exists a topological space \( Y \) and a function \( f : X \to Y \) that is not continuous.

(c) If \( Y \) is trivial, then every function \( f : X \to Y \), where \( X \) is any topological space, is continuous.

(d) If \( Y \) is discrete and contains at least two elements, then there exists a topological space \( X \) and a function \( f : X \to Y \) that is not continuous.

2. Show that the unit open ball \( B = \{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1 \} \) in \( \mathbb{R}^2 \) is homeomorphic to the open square \( C = (-1, 1) \times (-1, 1) \) using the usual topology.

3. Show that the unit open ball \( B = \{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1 \} \) in \( \mathbb{R}^2 \) is homeomorphic to the open right half plane \( H = (0, \infty) \times \mathbb{R} \) using the usual topology.

4. Let \( f, g : X \to Y \) be continuous functions between topological spaces, and assume that \( Y \) is Hausdorff. Prove that \( \{ x \in X : f(x) = g(x) \} \) is a closed subset of \( X \).

5. Let \( (X, T) \) be a compact topological space and let \( A, B \) are closed subsets of \( (X, T) \). Show that \( A \cup B \) is compact.

6. Let \( X = (0, 1) \) and let

\[
T = \{ A \subseteq \mathbb{R} \mid A = \emptyset \text{ or } A = (0, 1) \text{ or } A = (0, 1 - 1/n) \text{ for } n \geq 2 \}.
\]

Show that every open set \( A \) other than \( X \) is compact. Is \( X \) compact?

7. Let \( T \) be the co-countable topology on \( \mathbb{R} \), that is,

\[
T = \{ A \subseteq \mathbb{R} \mid A = \emptyset \text{ or } \mathbb{R} \setminus A \text{ is countable} \}.
\]

Is \([0, 1]\) compact in \((\mathbb{R}, T)\)? What are the compact sets in \((\mathbb{R}, T)\)?

8. Suppose that \( \{X_i\}_{i \in I} \) is a family of non-empty closed sets of a Hausdorff topological space \( X \), and that at least one of the \( X_i \) is compact. In addition, assume that the family has the property that for any two \( i, j \in I \) either \( X_i \subset X_j \) or \( X_j \subset X_i \). Show that \( \bigcap_{i \in I} X_i \neq \emptyset \).

9. Let \( (X, d) \) be a metric space and let \( \mathcal{H} \) be the collection of all non-empty compact subsets of \( X \). Define

\[
D(A, B) = \max \{ \sup \{ d(x, A) \mid x \in B \}, \sup \{ d(x, B) \mid x \in A \} \}.
\]

(a) Show that \( D \) defines a metric on \( \mathcal{H} \). The metric \( D \) is called the *Hausdorff metric*.

(b) Show that if \( (X, d) \) is complete, then \( (\mathcal{H}, D) \) is complete.

(b) Show that if \( (X, d) \) is totally bounded, then \( (\mathcal{H}, D) \) is totally bounded.

(b) Show that if \( (X, d) \) is compact, then \( (\mathcal{H}, D) \) is compact.