

Some solutions to Problem Set 1.

1. (a) d does not satisfy the first axiom since $d(-1, 1) = |(-1)^2 - (1)^2| = 0$.
 (b) d is a metric. (c) d is a metric. If $d(x, y) = |\arctan x - \arctan y| = 0$, then $\arctan x = \arctan y$, so that $x = y$ since $x \mapsto \arctan x$ is a one-to-one function. The remaining axioms are evident.

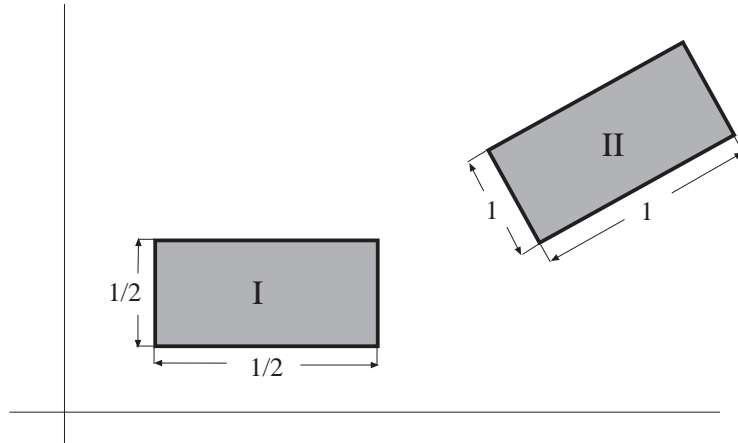
3. Note that $d(x, y)$ is half the number of horizontal and vertical steps needed to go from x to y . The first two axioms are trivially satisfied. To see that d satisfies the triangle inequality we argue by contradiction and assume that there are $x, y, z \in \mathbb{R}^2$ such that

$$d(x, z) > d(x, y) + d(y, z).$$

In view of the definition of d , this can happen if:

- (1) $d(x, z) > 0$ and $d(x, y) = d(y, z) = 0$.
- (2) $d(x, z) = 1$ and $d(x, y) + d(y, z) = 1/2$.

In the case (1), $x = y = z$ since $d(x, y) = d(y, z) = 0$. But this contradicts $d(x, z) > 0$. In the case (2), one of the numbers $d(x, y), d(y, z)$ is equal to 0 and the other is equal to $1/2$. Say $d(x, y) = 0$ and $d(y, z) = 1/2$. Hence $x = y$ and so $d(x, z) = 1/2$ contradicting $d(x, z) = 1$. The “area” of the rectangle I is equal to $1/2 \cdot 1/2 = 1/4$. The “area” of the rectangle II is equal to $1 \cdot 1 = 1$.



4. Clearly, $d_f(x, y) = 0$ if and only if $x = y$ and $d_f(x, y) = d_f(y, x)$. If x, y and $z \in X$, then $d(x, z) \leq d(x, y) + d(y, z)$ and

$$\begin{aligned} d_f(x, z) &= f(d(x, z)) \leq f(d(x, y) + d(y, z)) && \text{by (a)} \\ &\leq f(d(x, y)) + f(d(y, z)) && \text{by (c)} \\ &= d_f(x, y) + d_f(y, z) \end{aligned}$$

Properties (a)–(c) are clear for $f(t) = kt$. (Then $d_f = kd$ is the metric d rescaled by a factor $k > 0$). The derivatives of $f(t) = t^\alpha$ and $f(t) = \frac{t}{1+t}$ are positive for $t > 0$ so both functions are increasing. The property (b) is obvious for these two functions. If $0 < t \leq s$, then $(t + s)^{\alpha-1} \leq s^{\alpha-1} \leq t^{\alpha-1}$ (since $\alpha \leq 1$) and

$$(t + s)^\alpha = (t + s)(t + s)^{\alpha-1} \leq (t + s)s^{\alpha-1} \leq s^\alpha + t^\alpha,$$

hence $f(t+s) \leq f(t) + f(s)$. For $f(t) = \frac{t}{1+t}$ we have

$$\frac{t+s}{1+t+s} = \frac{t}{1+t+s} + \frac{s}{1+t+s} \leq \frac{t}{1+t} + \frac{s}{1+s}$$

so that $f(t+s) \leq f(t) + f(s)$.

6. If $d(x, y) = 0$, then $d_i(x_i, y_i) = 0$, $1 \leq i \leq n$, so that $x_i = y_i$ for $1 \leq i \leq n$. So $x = (x_1, \dots, x_n) = (y_1, \dots, y_n)$. Since $d_i(x_i, y_i) = d_i(y_i, x_i)$ for $1 \leq i \leq n$, we have $d(x, y) = d(y, x)$. Let $x, y, z \in \prod_{i=1}^n X_i$. Set $a_i = d_i(x_i, z_i)$, $b_i = d_i(x_i, y_i)$, and $c_i = d_i(y_i, z_i)$. We have $a_i \leq b_i + c_i$ for all $1 \leq i \leq n$, so

$$\left[\sum_{i=1}^n a_i^2 \right]^{1/2} \leq \left[\sum_{i=1}^n (b_i + c_i)^2 \right]^{1/2}.$$

Using Cauchy's inequality

$$\left[\sum_{i=1}^n (b_i + c_i)^2 \right]^{1/2} \leq \left[\sum_{i=1}^n b_i^2 \right]^{1/2} + \left[\sum_{i=1}^n c_i^2 \right]^{1/2},$$

hence

$$d(x, z) = \left[\sum_{i=1}^n a_i^2 \right]^{1/2} \leq \left[\sum_{i=1}^n b_i^2 \right]^{1/2} + \left[\sum_{i=1}^n c_i^2 \right]^{1/2} = d(x, y) + d(y, z),$$

and the triangle inequality follows.

8. First note that for all $x, y \in X$, $\frac{d_n(x_n, y_n)}{1+d_n(x_n, y_n)} < 1$ for all n , so

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{d_n(x_n, y_n)}{1+d_n(x_n, y_n)} < \sum_{n=1}^{\infty} \frac{1}{2^n} = 1,$$

and d is a well-defined function $d : X \times X \rightarrow \mathbb{R}$.

If $d(x, y) = 0$, then $\frac{d_n(x_n, y_n)}{1+d_n(x_n, y_n)} = 0$ for all n . Hence $d_n(x_n, y_n) = 0$ for all n implying $x_n = y_n$ for all n . So $x = y$. Clearly, $d(x, y) = d(y, x)$ since $d_n(x_n, y_n) = d_n(y_n, x_n)$. In view of the solution of (4), $\frac{d_n(a, b)}{1+d_n(a, b)}$, $a, b \in X_n$, is a metric on X_n and so

$$\begin{aligned} d(x, z) &= \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{d_n(x_n, z_n)}{1+d_n(x_n, z_n)} \leq \sum_{n=1}^{\infty} \frac{1}{2^n} \left[\frac{d_n(x_n, y_n)}{1+d_n(x_n, y_n)} + \frac{d_n(y_n, z_n)}{1+d_n(y_n, z_n)} \right] \\ &= \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{d_n(x_n, y_n)}{1+d_n(x_n, y_n)} + \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{d_n(y_n, z_n)}{1+d_n(y_n, z_n)} = d(x, y) + d(y, z). \end{aligned}$$

10. (a) For integers $n, m \geq 1$ we have $d(2n, 2m) = \left| \frac{1}{2n} - \frac{1}{2m} \right| = \frac{1}{2} \cdot \left| \frac{1}{n} - \frac{1}{m} \right| \leq \frac{1}{2}$. So $\text{diam}(P) \leq 1/2$. To see that $\text{diam}(P) = \frac{1}{2}$ note that $d(2n, 2) = \frac{1}{2} \left(1 - \frac{1}{n}\right) \rightarrow \frac{1}{2}$ as $n \rightarrow \infty$. For integers $n, m \geq 0$ we have $d(2n+1, 2m+1) = \left| \frac{1}{2n+1} - \frac{1}{2m+1} \right| \leq 1$. So $\text{diam}(\mathbb{N} \setminus P) \leq 1$. Since $d(2n+1, 1) = 1 - \frac{1}{2n+1} \rightarrow 1$ as $n \rightarrow \infty$, we have $\text{diam}(\mathbb{N} \setminus P) = 1$.

(b) $B(2n, \frac{1}{2n}) = \{m \in \mathbb{N} \mid \left| \frac{1}{2n} - \frac{1}{m} \right| < \frac{1}{2n}\} = \{m \in \mathbb{N}, n < m\}$.

$$B(n, \frac{1}{2n}) = \{m \in \mathbb{N} \mid \left| \frac{1}{n} - \frac{1}{m} \right| < \frac{1}{2n}\} = \{m \in \mathbb{N} \mid \frac{2n}{3} < m < 2n\}.$$