Some solutions to Problem Set 10.

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1. (a) Both H and L are closed in X since they are closed in $\mathbb{R}^2$. Since $H = X \setminus L$ and $L = X \setminus H$, they are also open in X. So X is disconnected.
(b) Every C_n is connected. Since all of the circles C_n, $n \in \mathbb{Z}$, have a common point $(0, 0)$, the union is connected.

2. Note that A is a connected subset of B, and the closure of A in B is $\overline{A} \cap B = B$. But the closure of a connected set is connected by a theorem from class, hence B is connected.

3. Let $Y = A \cup B$ and assume that $f : Y \to \{0, 1\}$ is continuous, where $\{0, 1\}$ is given the discrete topology. We will show that f is constant. Since A is connected and $f|A$ is continuous, $f|A$ is constant, say $f(A) = \{0\}$. Let $x \in \overline{A} \cap B$. Then $x \in \overline{A} \cap Y$, which is the closure of A in Y. Now since f is continuous, $f(x) \in \overline{f(A)} = \overline{\{0\}} = \{0\}$. Hence $f(x) = 0$. But B is also connected and $f|B$ is continuous, hence $f|B$ is constant, and since $x \in B$ we must have $f(B) = 0$. Hence $f = 0$ on all of $A \cup B$, and $A \cup B$ is connected.

4. Let X be a topological space with a cut point, say $p \in X$, and let $f : X \to Y$ be a homeomorphism. We claim that $f(p)$ is a cut point of Y. Since $X \setminus \{p\}$ is disconnected, there are two disjoint sets U and V such that U and V are open in $X \setminus \{p\}$, $U \cup V = X \setminus \{p\}$. Then $g = f|_{X \setminus \{p\}} : X \setminus \{p\} \to Y \setminus \{f(p)\}$ is a homeomorphism, and so $g(U)$ and $g(V)$ are open in $Y \setminus \{f(p)\}$. But $g(U) \cap g(V) = \emptyset$, and $Y \setminus \{f(p)\} = g(U) \cup g(V)$. Hence $Y \setminus \{f(p)\}$ is disconnected, and $f(p)$ is a cut point of Y.

5. Consider (a, b) and $(a, b]$. Assume that $f : (a, b] \to (a, b)$ is a homeomorphism. Let $c = f(b)$. Then c is a cut-point of (a, b). So by the previous problem, $f^{-1}(c) = b$ is a cut point of $(a, b]$, that is $(a, b] \setminus \{b\} = (a, b)$ is disconnected, contradiction.
The proof for (a, b) and $[a, b]$ is similar.
Consider $(a, b]$ and $[a, b]$. Assume that $f : [a, b] \to (a, b]$ is a homeomorphism. Since f is one-one, one of the points $f(a), f(b)$ belongs to (a, b). Say $f(a) \in (a, b)$. Then $f(a)$ is a cut point of (a, b) and so $f^{-1}(f(a))$ is a cut point of $[a, b]$. But $[a, b] \setminus \{f^{-1}(f(a))\} = [a, b] \setminus \{a\} = (a, b]$ which is connected, contradiction. Hence $(a, b]$, and $[a, b]$ are not homeomorphic.

6. First notice that $\mathbb{R}^2 \setminus \{a\}$ is path connected since any two points $x, y \in \mathbb{R}^2 \setminus \{a\}$ can be joined by a continuous path. So $\mathbb{R}^2 \setminus \{a\}$ is connected. Assume now that $\mathbb{R}$ and $\mathbb{R}^2$ are homeomorphic. Then there is a homeomorphism $f : \mathbb{R}^2 \to \mathbb{R}$. Denote $x = f(0, 0)$. The map $g = f|_{\mathbb{R}^2 \setminus \{(0,0)\}} : \mathbb{R}^2 \setminus \{(0, 0)\} \to \mathbb{R} \setminus \{x\}$ is a homeomorphism. Since, by above, $\mathbb{R}^2 \setminus \{(0, 0)\}$ is connected, $f(\mathbb{R}^2 \setminus \{(0, 0)\}) = \mathbb{R} \setminus \{x\}$ is connected, contradiction.

7. Define $g : S^1 \to \mathbb{R}$ by $g(x) = f(x) - f(-x)$. Then g is continuous and $g(-x) = f(-x) - f(x) = -g(x)$, so if $g(x_0) \geq 0$, say, then $g(-x_0) \leq 0$. Since since S^1 is connected, the intermediate value theorem shows that there exists $x \in S^1$ such that $g(x) = 0$. But this means that $f(x) = f(-x)$ as required.

8. Take two distinct points $x, y \in \mathbb{R}^2 \backslash A$. Since A is countable and there uncountably many lines passing through x, there are uncountably many lines passing through x and containing no point of A. Take one of such lines, say $L \subseteq X \setminus A$. If the point y belongs to L, then $\alpha(t) = (1-t)x + ty \in L$ for $t \in [0,1]$, $\alpha(0) = x$, $\alpha(1) = y$. Hence x and y can be connected by a continuous path in $\mathbb{R}^2 \setminus A$. If $y \notin L$, then there are uncountably many lines through y which don't intersect A and are, in addition, not parallel to L. Pick one of them, say K. Then $y \in K$, $K \subseteq \mathbb{R}^2 \setminus A$, and K intersects L at a point z. Now let $\alpha(t) = (1-t)x + tz$, $\beta(t) = (1-t)z + ty$ for $t \in [0,1]$. Then $\alpha(t), \beta(t) \in \mathbb{R}^2 \setminus A$, $\alpha(0) = x, \alpha(1) = z, \beta(0) = z$, and $\beta(1) = y$. Finally, define

$$\gamma(t) = \begin{cases} \alpha(2t) & 0 \leqslant t \leqslant 1/2, \\ \beta(2t-1) & 1/2 \leqslant t \leqslant 1. \end{cases}$$

Then γ is continuous by the gluing lemma, and is a path in $\mathbb{R}^2 \setminus A$ joining x with y.

9. Fix a point $x_0 \in A$ and let U be the collection of all $x \in A$ such that there is a path $\alpha : [0,1] \to A$ from x_0 to x, that is $\alpha(0) = x_0$ and $\alpha(1) = x$. The set U is not empty since $x_0 \in U$, and we claim that U is open. Indeed, let $x \in U$. Since A is open, there is a ball $B(x,r)$ such that $B(x,r) \subseteq A$. We will show that $B(x,r) \subset U$. Take $y \in B(x,r)$. Then there is a continuous path $\beta(t) = (1-t)x + ty \in B(x,r)$, $0 \leqslant t \leqslant 1$, joining x with y. Since there is a continuous path $\alpha : [0,1] \to A$ such that $\alpha(0) = x_0$ and $\alpha(1) = x$, the path defined by

$$\gamma(t) = \begin{cases} \alpha(2t) & \text{for } 0 \leqslant t \leqslant 1/2 \\ \beta(2t-1) & \text{for } 1/2 \leqslant t \leqslant 1 \end{cases}$$

satisfies $\gamma(0) = x_0$, $\gamma(1) = y$, $\gamma(t) \in A$ for all $t \in [0,1]$. Further, γ is continuous by the gluing lemma. So $y \in U$, and $B(x,r) \subseteq U$, and U is open.
Now $V = A \setminus U$ be the collection of points $x \in A$ which cannot be joined to x_0 by a continuous path in A. Then V is open. To see this assume that $x \in V$. Then $B(x,r) \subseteq A$ for some $r > 0$. Any point $y \in B(x,r)$ can be connected by a continuous path with x, just take $\beta(t) = (1-t)y + tx$. If y can be connected to x_0 by a continuous path, then x_0 can be connected by a continuous path with x, contradiction. Hence $B(x,r) \subset V$. Since $A = U \cup V$ and $U \cap V = \emptyset$, and A is connected, one of these sets has to be empty. Since $x_0 \in U$, then $V = \emptyset$. So $A = U$, and the result follows.

10. Assume not. Then there is a continuous surjective function $f : X \to \{0,1\}$. We consider $\{0,1\}$ with the discrete metric d. Since (X, d_X) is compact, then f is uniformly continuous. Let $0 < \varepsilon < 1$. Then there is $\delta > 0$ such that if $d_X(x,y) < \delta$, then $d(f(x), f(y)) < \varepsilon$. Since f is a surjection, there are x and y in X such that $f(x) = 0$ and $f(y) = 1$. Since X is chain connected, there is a finite set of points $x_0 = x, x_1, \ldots, x_{n-1}, x_n = y$ such that $d_X(x_i, x_{i+1}) < \delta$ for $0 \leqslant i \leqslant n-1$. Thus, $d(f(x_i), f(x_{i+1})) < \varepsilon$ for $0 \leqslant i \leqslant n-1$. Since d is the discrete metric and $\varepsilon < 1$, $f(x_i) = f(x_{i+1})$ for $0 \leqslant i \leqslant n-1$. This means that $f(x) = f(x_0) = f(x_1) = \cdots = f(x_{n-1}) = f(x_n) = f(y)$, contradicting $f(x) = 0$ and $f(y) = 1$. So X is connected.