Some solutions to Problem Set 10.

1. (a) Both $H$ and $L$ are closed in $X$ since they are closed in $\mathbb{R}^2$. Since $H = X \setminus L$ and $L = X \setminus H$, they are also open in $X$. So $X$ is disconnected.
(b) Every $C_n$ is connected. Since all of the circles $C_n$, $n \in \mathbb{Z}$, have a common point $(0,0)$, the union is connected.

2. Note that $A$ is a connected subset of $B$, and the closure of $A$ in $B$ is $\overline{A} \cap B = B$. But the closure of a connected set is connected by a theorem from class, hence $B$ is connected.

3. Let $Y = A \cup B$ and assume that $f : Y \to \{0,1\}$ is continuous, where $\{0,1\}$ is given the discrete topology. We will show that $f$ is constant. Since $A$ is connected and $f|A$ is continuous, $f|A$ is constant, say $f(A) = \{0\}$. Let $x \in \overline{A} \cap B$. Then $x \in \overline{A} \cap Y$, which is the closure of $A$ in $Y$. Now since $f$ is continuous, $f(x) \in f(A) = \{0\} = \{0\}$. Hence $f(x) = 0$. But $B$ is also connected and $f|B$ is continuous, hence $f|B$ is constant, and since $x \in B$ we must have $f(B) = 0$. Hence $f = 0$ on all of $A \cup B$, and $A \cup B$ is connected.

4. Let $X$ be a topological space with a cut point, say $p \in X$, and let $f : X \to Y$ be a homeomorphism. We claim that $f(p)$ is a cut point of $Y$. Since $X \setminus \{p\}$ is disconnected, there are two disjoint sets $U$ and $V$ such that $U$ and $V$ are open in $X \setminus \{p\}$, $U \cup V = X \setminus \{p\}$. Then $g = f|_{X\setminus\{p\}} : X \setminus \{p\} \to Y \setminus \{f(p)\}$ is a homeomorphism, and so $g(U)$ and $g(V)$ are open in $Y \setminus \{f(p)\}$. But $g(U) \cap g(V) = \emptyset$, and $Y \setminus \{f(p)\} = g(U) \cup g(V)$. Hence $Y \setminus \{f(p)\}$ is disconnected, and $f(p)$ is a cut point of $Y$.

5. Consider $(a,b)$ and $(a,b)$. Assume that $f : (a,b) \to (a,b)$ is a homeomorphism. Let $c = f(b)$. Then $c$ is a cut-point of $(a,b)$. So by the previous problem, $f^{-1}(c) = b$ is a cut point of $(a,b)$, that is $(a,b) \setminus \{b\} = (a,b)$ is disconnected, contradiction. The proof for $(a,b)$ and $[a,b]$ is similar.

Consider $(a,b)$ and $[a,b]$. Assume that $f : [a,b] \to (a,b)$ is a homeomorphism. Since $f$ is one-one, one of the points $f(a)$, $f(b)$ belongs to $(a,b)$. Say $f(a) \in (a,b)$. Then $f(a)$ is a cut point of $(a,b)$ and so $f^{-1}(f(a))$ is a cut point of $[a,b]$. But $[a,b] \setminus \{f^{-1}(f(a))\} = [a,b] \setminus \{a\} = (a,b]$ which is connected, contradiction. Hence $(a,b)$, and $[a,b]$ are not homeomorphic.

6. First notice that $\mathbb{R}^2 \setminus \{a\}$ is path connected since any two points $x, y \in \mathbb{R}^2 \setminus \{a\}$ can be joined by a continuous path. So $\mathbb{R}^2 \setminus \{a\}$ is connected. Assume now that $\mathbb{R}$ and $\mathbb{R}^2$ are homeomorphic. Then there is a homeomorphism $f : \mathbb{R}^2 \to \mathbb{R}$. Denote $x = f(0,0)$. The map $g = f|_{\mathbb{R}^2 \setminus \{(0,0)\}} : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R} \setminus \{x\}$ is a homeomorphism. Since, by above, $\mathbb{R}^2 \setminus \{(0,0)\}$ is connected, $f(\mathbb{R}^2 \setminus \{(0,0)\}) = \mathbb{R} \setminus \{x\}$ is connected, contradiction.

7. Define $g : S^1 \to \mathbb{R}$ by $g(x) = f(x) - f(-x)$. Then $g$ is continuous and $g(-x) = f(-x) - f(x) = -g(x)$, so if $g(x_0) \geq 0$, say, then $g(-x_0) \leq 0$. Since since $S^1$ is connected, the intermediate value theorem shows that there exists $x \in S^1$ such that $g(x) = 0$. But this means that $f(x) = f(-x)$ as required.
8. Take two distinct points \( x, y \in \mathbb{R}^2 \setminus A \). Since \( A \) is countable and there uncountably many lines passing through \( x \), there are uncountably many lines passing through \( x \) and containing no point of \( A \). Take one of such lines, say \( L \subseteq X \setminus A \). If the point \( y \) belongs to \( L \), then \( \alpha(t) = (1 - t)x + ty \in L \) for \( t \in [0, 1] \), \( \alpha(0) = x \), \( \alpha(1) = y \). Hence \( x \) and \( y \) can be connected by a continuous path in \( \mathbb{R}^2 \setminus A \). If \( y \notin L \), then there are uncountably many lines through \( y \) which don’t intersect \( A \) and are, in addition, not parallel to \( L \). Pick one of them, say \( K \). Then \( y \in K, K \subseteq \mathbb{R}^2 \setminus A \), and \( K \) intersects \( L \) at a point \( z \). Now let \( \alpha(t) = (1 - t)x + tz, \beta(t) = (1 - t)z + ty \) for \( t \in [0, 1] \). Then \( \alpha(t), \beta(t) \in \mathbb{R}^2 \setminus A, \alpha(0) = x, \alpha(1) = z, \beta(0) = z, \) and \( \beta(1) = y \). Finally, define

\[
\gamma(t) = \begin{cases} 
\alpha(2t) & 0 \leq t \leq 1/2, \\
\beta(2t - 1) & 1/2 \leq t \leq 1.
\end{cases}
\]

Then \( \gamma \) is continuous by the gluing lemma, and is a path in \( \mathbb{R}^2 \setminus A \) joining \( x \) with \( y \).

9. Fix a point \( x_0 \in A \) and let \( U \) be the collection of all \( x \in A \) such that there is a path \( \alpha : [0, 1] \to A \) from \( x_0 \) to \( x \), that is \( \alpha(0) = x_0 \) and \( \alpha(1) = x \). The set \( U \) is not empty since \( x_0 \in U \), and we claim that \( U \) is open. Indeed, let \( x \in U \). Since \( A \) is open, there is a ball \( B(x, r) \) such that \( B(x, r) \subseteq A \). We will show that \( B(x, r) \subseteq U \).

Take \( y \in B(x, r) \). Then there is a continuous path \( \beta(t) = (1 - t)x + ty \in B(x, r), 0 \leq t \leq 1 \), joining \( x \) with \( y \). Since there is a continuous path \( \alpha : [0, 1] \to A \) such that \( \alpha(0) = x_0 \) and \( \alpha(1) = x \), the path defined by

\[
\gamma(t) = \begin{cases} 
\alpha(2t) & 0 \leq t \leq 1/2, \\
\beta(2t - 1) & 1/2 \leq t \leq 1.
\end{cases}
\]

satisfies \( \gamma(0) = x_0, \gamma(1) = y, \gamma(t) \in A \) for all \( t \in [0, 1] \). Further, \( \gamma \) is continuous by the gluing lemma. So \( y \in U \), and \( B(x, r) \subseteq U \), and \( U \) is open.

Now \( V = A \setminus U \) be the collection of points \( x \in A \) which cannot be joined to \( x_0 \) by a continuous path in \( A \). Then \( V \) is open. To see this assume that \( x \in V \). Then \( B(x, r) \subseteq A \) for some \( r > 0 \). Any point \( y \in B(x, r) \) can be connected by a continuous path with \( x \), just take \( \beta(t) = (1 - t)y + t \). If \( y \) can be connected to \( x_0 \) by a continuous path, then \( x_0 \) can be connected by a continuous path with \( x \), contradiction. Hence \( B(x, r) \subseteq V \). Since \( A = U \cup V \) and \( U \cap V = \emptyset \), and \( A \) is connected, one of these sets has to be empty. Since \( x_0 \in U \), then \( V = \emptyset \). So \( A = U \), and the result follows.

10. Assume not. Then there is a continuous surjective function \( f : X \to \{0, 1\} \). We consider \( \{0, 1\} \) with the discrete metric \( d \). Since \( (X, d_X) \) is compact, then \( f \) is uniformly continuous. Let \( 0 < \varepsilon < 1 \). Then there is \( \delta > 0 \) such that if \( d_X(x, y) < \delta \), then \( d(f(x), f(y)) < \varepsilon \). Since \( f \) is a surjection, there are \( x \) and \( y \) in \( X \) such that \( f(x) = 0 \) and \( f(y) = 1 \). Since \( X \) is chain connected, there is a finite set of points \( x_0 = x, x_1, \ldots, x_{n-1}, x_n = y \) such that \( d_X(x_i, x_{i+1}) < \delta \) for \( 0 \leq i \leq n - 1 \). Thus, \( d(f(x_i), f(x_{i+1})) < \varepsilon \) for \( 0 \leq i \leq n - 1 \). Since \( d \) is the discrete metric and \( \varepsilon < 1 \), \( f(x_i) = f(x_{i+1}) \) for \( 0 \leq i \leq n - 1 \). This means that \( f(x) = f(x_0) = f(x_1) = \cdots = f(x_{n-1}) = f(x_n) = f(y) \), contradicting \( f(x) = 0 \) and \( f(y) = 1 \). So \( X \) is connected.