Some solutions to Problem Set 11.

---

1. (a) $X \setminus A$ is open in X since A is closed in X, and Y is open in Y. Hence $(X \setminus A) \times Y$ is open in $X \times Y$, and $A \times Y$ is closed in $X \times Y$. Similarly $X \times (Y \setminus B)$ is open in $X \times Y$, so $X \times B$ is closed in $X \times Y$. Hence $A \times B = (A \times Y) \cap (X \times B)$ is closed in $X \times Y$.
(b) Let $W = \{(x, y) \in \mathbb{R}^2 : xy = 1\}$. Then W is closed in $\mathbb{R} \times \mathbb{R}$ since it is the preimage $f^{-1}(\{1\})$ of a closed set under the continuous map $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = xy$. But $\pi_1(W) = \mathbb{R} - \{0\}$ is not closed in $\mathbb{R}$.

2. Assume X is Hausdorff and let $(x, y) \in X \times X \setminus \Delta$. Then $x \neq y \in X$ so there exist disjoint open sets U, V with $x \in U$ and $y \in V$. If $(a, b) \in U \times V$ then $a \neq b$, hence $U \times V \subset X \times X \setminus \Delta$. But $W(x, y) = U \times V$ is open in the product topology, hence $X \times X \setminus \Delta = \bigcup\{W(x, y) : (x, y) \in X \times X \setminus \Delta\}$ is open and Δ is closed in $X \times X$.
Conversely, assume Δ is closed, so $X \times X \setminus \Delta$ is open. Now the product topology on $X \times X$ has a basis consisting of products $U \times V$ where U and V are open in X. Given $x, y \in X$ with $x \neq y$, we have $(x, y) \in X \times X \setminus \Delta$, hence there exist open sets U and V in X such that $(x, y) \in U \times V \subset X \times X \setminus \Delta$. Then $x \in U, y \in V$ and $U \cap V = \emptyset$. Hence X is Hausdorff.

3. For (c) and (d) consider the projection $\pi_j : X \to X_j$. Since X is compact (connected) and π_j is continuous, then $\pi_j(X)$ is compact (connected). And since $\pi_j(X) = X_j$, the space X_j is compact (connected).
For (a) and (b), fix points $x_i \in X_i$ for $1 \leq i \leq n$ and define $f_j : X_j \to X$ by $f_j(x) = (x_1, \ldots, x_{j-1}, x, x_{j+1}, \ldots, x_n)$. This is a homeomorphism X_j onto $f_j(X_j)$ in X, whose inverse is the restriction of the projection π_j from $f_j(X_j)$ to X_j.
Assume that X is normal and let A, B be disjoint closed subsets of X_j. Then $f_j(A), f_j(B)$ are disjoint closed subsets of $f_j(X_j)$, and $f_j(X_j)$ is closed in X by Q1 (and induction). Hence $f_j(A), f_j(B)$ are disjoint closed subsets of X, so there are disjoint open sets U and V in X such that $f_j(A) \subset U$ and $f_j(B) \subset V$. Then $U' = U \cap f_j(X_j)$ and $V' = V \cap f_j(X_j)$ are disjoint open subsets of $f_j(X_j)$, and $\pi_j(U')$, $\pi_j(V')$ are disjoint open subsets of X_j containing A, B respectively. Hence X_j is normal. (a) is proved in the same way.

4. (a) $\{f_n\}$ is pointwise bounded since $|f_n(x)| \leqslant |x| + 1$ for all n, but it is not equicontinuous. Indeed, take $x_n = \frac{\pi}{2n}$ and $y_n = \frac{\pi}{n}$. Then $|x_n - y_n| = \frac{\pi}{2n} \to 0$ as $n \to \infty$ but $|f_n(x_n) - f(y_n)| \geq |\sin(nx_n) - \sin(ny_n)| - |x_n - y_n| = 1 - \frac{\pi}{2n} \to 1$.
(b) $\{g_n\}$ is not pointwise bounded since $|g_n(x)| \geq n - 1$ but is equicontinuous. Indeed, $|g_n(x) - g_n(y)| = |\sin x - \sin y| \leqslant |x - y|$ by the mean value theorem. Hence given $\varepsilon > 0$ we can choose $\delta = \varepsilon$, then $|x - y| < \delta$ implies $|g_n(x) - g_n(y)| < \epsilon$ for all $n \in \mathbb{N}$.

(c) $\{h_n\}$ is pointwise bounded since $|h_n(x)| \leqslant 1$ if $|x| \leqslant 1$ and $|h_n(x)| = |x|^{1/n} \leqslant |x|$ if $|x| \geq 1$. If $\{h_n\}$ is equicontinuous then for $\varepsilon = 1/2$ there is $\delta > 0$ such that $|0 - x| = |x| < \delta$ implies $|x|^{1/n} < 1/2$ for all $n \in \mathbb{N}$. But if we take $x_n = 1/n$, then $h_n(x_n) = \frac{1}{\sqrt[n]{n}} \to 1$, contradiction.

(d) $\{k_n\}$ is pointwise bounded. Indeed, $|\sin(x/n)| \leqslant |x|/n$. So $|k_n(x)| \leqslant |x|$. Moreover, $|k_n(x) - k_n(y)| = n|\sin(x/n) - \sin(y/n)| \leqslant n \cdot |(x/n) - (y/n)| = |x - y|$. Hence the family $\{k_n\}$ is equicontinuous.

5. The family $\mathcal{F}$ is equicontinuous and bounded. To see the equicontinuity let $\varepsilon > 0$ and take $0 < \delta < \varepsilon/M$. Then for any $x, y \in [a, b]$ such that $|x - y| < \delta$ and all $f \in \mathcal{F}$, the mean value theorem gives:

$$|f(x) - f(y)| = |f'(\eta)| \cdot |x - y|$$

for some η between x and y. Hence

$$|f(x) - f(y)| \leqslant M|x - y| \leqslant M \cdot (\varepsilon/M) = \varepsilon.$$

Moreover, for all $x \in [a, b]$ and all $f \in \mathcal{F}$,

$$|f(x)| \leqslant |f(x_0)| + |f(x_0) - f(x)| \leqslant M + M(b - a).$$

So $\mathcal{F}$ is bounded. By the Ascoli-Arzela theorem every sequence in $\mathcal{F}$ has a uniformly convergent subsequence, and $\overline{\mathcal{F}}$ is compact in $C[a, b]$.

6. (a) Consider the space $C(X, X)$ with the uniform metric d_∞. By the Ascoli-Arzela theorem some subsequence of $\{I_n\}$ converges to a continuous map $f : X \to X$. We have to show that f is an isometry. Without loss of generality we assume that $d_\infty(I_n, f) \to 0$. If $x, y \in X$, then

$$d(f(x), f(y)) = \lim_n (I_n(x), I_n(y)) = \lim_n d(x, y) = d(x, y).$$

So f preserves distance, in particular, it is injective. Since X is compact and f preserves distance, $f(X) = X$ (this is the content of a problem on one of the previous problem sheets). Thus, f is an isometry from X onto X.
(b) Let $\{I_{n_k}\}$ be any subsequence of $\{I_n\}$. Then $\{I_{n_k}^{-1}\}$ is a sequence of isometries so by the above it has a subsequence $\{I_{n_{k_l}}^{-1}\}$ converging to some isometry g. Now if $f_n, g_n : X \to X$ are sequences of isometries converging in $C(X, X)$ to f, g respectively, then $f_n \circ g_n \to f \circ g$ in $C(X, X)$, since

$$\begin{aligned} d(f_n(g_n(x)), f(g(x))) &\leq d(f_n(g_n(x)), f_n(g(x))) + d(f_n(g(x)), f(g(x))) \\ &= d(g_n(x), g(x)) + d(f_n(g(x)), f(g(x))) \\ &\leq d_\infty(g_n, g) + d_\infty(f_n, f) \to 0 \text{ as } n \to \infty. \end{aligned}$$

Thus

$$\mathrm{id} = I_{n_{k_l}} \circ I_{n_{k_l}}^{-1} \to f \circ g \quad \text{and} \quad \mathrm{id} = I_{n_{k_l}}^{-1} \circ I_{n_{k_l}} \to g \circ f.$$

So $f \circ g = \mathrm{id}$ and $g \circ f = \mathrm{id}$, that is, $g = f^{-1}$. Since this holds for any subsequence $\{I_{n_k}\}$ of $\{I_n\}$, it follows that $I_n^{-1} \to I^{-1}$ provided that $I_n \to I$ in $C(X, X)$.
(c) This follows from (a) since any 3 by 3 orthogonal matrix A defines an isometry $A : S^2 \to S^2$ and S^2 is compact.

6. (a) For disjoint closed non-empty sets A and $B \subset X$ define

$$f(x) = \frac{d(x, A)}{d(x, A) + d(x, B)}.$$

The function is well-defined since $d(x, A) + d(x, B) > 0$ for all $x \in X$. (If $d(x, A) + d(x, B) = 0$, then $d(x, A) = 0 = d(x, B)$ and since A, B are closed, $x \in A$ and $x \in B$, contradicting $A \cap B = \emptyset$.) Moreover, f is continuous since $x \mapsto d(x, A)$ and $x \mapsto d(x, B)$ are continuous. The function f is equal to 0 if and only if $d(x, A) = 0$ if and only if $x \in A$ since A is closed. Similarly, $f(x) = 1$ if and only if $d(x, B) = 0$ if and only if $x \in B$.

(b) Let C be a closed subset of $\mathbb{R}$ and let $f : C \to \mathbb{R}$ be a bounded continuous function. The set $\mathbb{R} \setminus C$ is an open subset of $\mathbb{R}$ and so it can be represented as a countable union of disjoint open intervals, $\mathbb{R} \setminus C = \bigcup_{n\in\mathbb{N}}(a_n, b_n)$. (See notes, p 48.) The points $a_n, b_n \in C$ if the interval (a_n, b_n) is bounded; if $(a_n, b_n) = (-\infty, b_n)$, then $b_n \in C$; and if $(a_n, b_n) = (a_n, \infty)$, then $a_n \in C$. So define $h : \mathbb{R} \to \mathbb{R}$ as follows. We set $h(x) = f(x)$ for $x \in C$. If $(a_n, b_n) = (-\infty, b_n)$ we set $h(x) = f(b_n)$ for $x \in (-\infty, b_n)$. If $(a_n, b_n) = (a_n, \infty)$, we set $h(x) = f(a_n)$ for $x \in (a_n, \infty)$. If (a_n, b_n) is a bounded interval we set

$$h(x) = \frac{f(b_n) - f(a_n)}{b_n - a_n} \cdot (x - a_n) + f(a_n), \quad x \in (a_n, b_n).$$

Then h is a bounded continuous function such that $h(x) = f(x)$ for $x \in C$.

(c) Assume that X is the Hausdorff space on which Tietze's extension theorem holds. Take two non-empty disjoint closed subsets A, B of X. The set $A \cup B$ is closed in X. Then the function $f : A \cup B \to \mathbb{R}$ defined by $f(x) = 0$ if $x \in A$ and $f(x) = 1$ if $x \in B$ is continuous. Hence there is a continuous function $h : X \to \mathbb{R}$ such that $h = f$ on $A \cup B$. Set $U = f^{-1}((-\infty, 1/3))$ and $V = f^{-1}((2/3, \infty))$. These are open disjoint sets and satisfy $A \subset U$ and $B \subset V$. Hence X is normal as claimed. The converse is just Tietze's extension theorem.