Some solutions to Problem Set 2.

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1. Here is one interpretation of the metrics: d_M gives the shortest distance between two points if you can only travel vertically and along the x_1-axis; d_K gives the shortest distance if you can only travel along straight lines through the origin.

(b) The sequence (x_n) does not converge in $(\mathbb{R}^2, d_M)$ since if $x = (a,b) \in \mathbb{R}^2$, then $d_M(x_n, x) = |\frac{n}{n+1} - a| + \frac{n}{n+1} + |b| \geq \frac{1}{2}$ for all $n \geq 1$ with $\frac{n}{n+1} \neq a$. However, (x_n) converges to $x = (1,1)$ with respect to the metric d_K since $d_K(x_n, x) = \big\|(\frac{n}{n+1} - 1, \frac{n}{n+1} - 1)\big\| = \frac{\sqrt{2}}{n+1} \to 0$ as $n \to \infty$.

(c) The sequence (x_n) converges to $x = (0,0)$ in $(\mathbb{R}^2, d_M)$ since $d_M(x_n, x) = |\frac{1}{n} - 0| + (\sqrt{n+1} - \sqrt{n}) + |0| = \frac{1}{n} + \frac{1}{\sqrt{n+1} + \sqrt{n}} \to 0$ as $n \to \infty$. It also converges to $x = (0,0)$ in $(\mathbb{R}^2, d_K)$ since

$$d_K(x_n, x) = \big\|(\frac{1}{n}, \sqrt{n+1} - \sqrt{n})\big\| = \sqrt{\frac{1}{n^2} + \frac{1}{(\sqrt{n+1} + \sqrt{n})^2}} \leqslant \sqrt{2/n} \to 0.$$

2. Using the triangle inequality twice we have $d(x_n, y_n) \leq d(x_n, x) + d(x,y) + d(y, y_n)$, hence $d(x_n, y_n) - d(x,y) \leq d(x_n, x) + d(y_n, y)$. Similarly, we obtain $d(x_n, y_n) - d(x,y) \leq d(x_n, x) + d(y_n, y)$, hence $|d(x_n, y_n) - d(x,y)| \leq d(x_n, x) + d(y_n, y)$. As $n \to \infty$, we have $d(x_n, x) \to 0$ and $d(y_n, y) \to 0$, hence $|d(x_n, y_n) - d(x,y)| \to 0$, i.e. $d(x_n, y_n) \to d(x,y)$.

3. (a) Consider $\mathbb{R}$ with the metrics $d(x,y) = |x - y|$ and $\overline{d}(x,y) = \frac{|x-y|}{1+|x-y|}$. Then d and $\overline{d}$ are equivalent. However, they are not Lipschitz equivalent since if $x_n = n$ and $x = 0$, then $d(x_n, x) = n$ and $\overline{d}(x_n, x) = \frac{n}{1+n} < 1$ and so there is no constant C such that $d(x_n, x) \leqslant C \cdot \overline{d}(x_n, x)$ for all n.

(b) Define $d_\infty(x,y) = \max\{|x_1 - y_1|, \ldots, |x_n - y_n|\}$ for $x, y \in \mathbb{R}^n$. Note that for any $r \geq 1$,

$$d_\infty(x,y) \leqslant d_r(x,y) \leqslant n^{1/r} d_\infty(x,y).$$

Hence if $q > p \geq 1$, then

$$d_q(x,y) \leqslant n^{1/q} d_\infty(x,y) \leqslant n^{1/q} d_p(x,y)$$

and

$$d_p(x,y) \leqslant n^{1/p} d_\infty(x,y) \leqslant n^{1/p} d_q(x,y).$$

Combining the above inequalities one gets

$$n^{-1/q} d_q(x,y) \leqslant d_p(x,y) \leqslant n^{1/p} d_q(x,y).$$

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(a) A is open in X but not in $\mathbb{R}$, A is not closed in both spaces

(b) B is open in X, not open in $\mathbb{R}$, not closed in X and not closed in $\mathbb{R}$

(c) C is not open and not closed in X. It is not open and not closed in $\mathbb{R}$.

(d) D is not open in both spaces, and closed in both spaces.

(e) E open in both spaces, not closed in both spaces.

5. A is neither open nor closed, B is closed, C is neither open nor closed, D is neither open nor closed, E neither open nor closed.

6.

(a) $A^\circ = A$, $\overline{A} = \{(x,y) |\ x \geq 0\}$, $\partial A = \{(x,y) |\ x = 0, y \in \mathbb{R}\} \cup \{(x,y) |\ x \geq 0, y = 0\}$

(b) $B^\circ = \emptyset$, $B = \overline{B}$, $\partial B = B$

(c) $C^\circ = A$, $\overline{C} = \{(x,y) \in \mathbb{R}^2 |\ x \geq 0\}$, $\partial C = \{(x,y) | x = 0, y \in \mathbb{R}\} \cup \{(x,y) \in \mathbb{R}^2 | x \geq 0,\ y = 0\}$

(d) $D^\circ = \emptyset$, $\overline{D} = \mathbb{R}^2$, $\partial D = \mathbb{R}^2$.

(e) $F^\circ = \{(x,y) | x \neq 0 \text{ and } y < 1/x\}$, $\overline{F} = F \cup \{(x,y) |\ x = 0, y \in \mathbb{R}\}$, $\partial F = \{(x,y) \mid x \neq 0 \text{ and } y = 1/x\} \cup \{(x,y) |\ x = 0, y \in \mathbb{R}\}$.

7. The answer to both questions is No!. For example consider $A = \mathbb{Q}$ in $\mathbb{R}$ equipped with the usual metric. Then $A^\circ = \emptyset$ but $\overline{A} = \mathbb{R}$ so that $(\overline{A})^\circ = \mathbb{R}^\circ = \mathbb{R}$. Also $\overline{A^\circ} = \emptyset$.

8. • $\bigcup_{i \in I} A_i^\circ \subset \left(\bigcup_{i \in I} A_i\right)^\circ$: Let $x \in \bigcup_{i \in I} A_i^\circ$. Then $x \in A_i^\circ$ for some i. Hence $B(x,r) \subset A_i$ for some $r > 0$, and so $B(x,r) \subset \bigcup_{i \in I} A_i$. Consequently, x is an interior point of $\bigcup_{i \in I} A_i$.

• $\overline{\bigcap_{i \in I} A_i} \subset \bigcap_{i \in I} \overline{A_i}$: Let $x \in \overline{\bigcap_{i \in I} A_i}$. So for every $B(x,r) \cap \bigcap_{i \in I} A_i \neq \emptyset$ which means that $B(x,r) \cap A_i \neq \emptyset$ for all $i \in I$. Hence x is an adherent point of A_i for all $i \in I$, that is, $x \in \overline{A_i}$ for all $i \in I$. Therefore, $x \in \bigcap_{i \in I} \overline{A_i}$.

• $\left(\bigcap_{i \in I} A_i\right)^\circ \subset \bigcap_{i \in I} A_i^\circ$: Let $x \in \left(\bigcap_{i \in I} A_i\right)^\circ$. So $B(x,r) \subset \bigcap_{i \in I} A_i$. Hence $B(x,r) \subset A_i$ for all $i \in I$, that is $x \in A_i^\circ$ for all $i \in I$. Therefore, $x \in \bigcap_{i \in I} A_i^\circ$.

• $\bigcup_{i \in I} \overline{A_i} \subset \overline{\bigcup_{i \in I} A_i}$: Let $x \in \bigcup_{i \in I} \overline{A_i}$. Then $x \in \overline{A_j}$ for some $j \in I$, and so $B(x,r) \cap A_j \neq \emptyset$. Consequently, $B(x,r) \cap \left(\bigcup_{i \in I} A_i\right) \neq \emptyset$. This means that x is an adherent point of $\bigcup_{i \in I} A_i$, i.e., $x \in \overline{\bigcup_{i \in I} A_i}$.

9. (a) Since $A \subset \overline{A}$ and $\partial A \subset \overline{A}$, $A \cup \partial A \subset \overline{A}$. Conversely, if $x \in \overline{A}$ and $x \notin A$ then $x \in \overline{X \setminus A}$ so that $x \in \partial A$. Consequently, $\overline{A} \subset A \cup \partial A$.
(b) If $x \in \partial A$, then $x \in \overline{A}$ and $x \in \overline{X \setminus A}$. Since $x \in \overline{X \setminus A}$, for every $r > 0$, $B(x,r) \cap (X \setminus A) \neq \emptyset$, implying that $x \notin A^0$. Hence $x \in \overline{A} \setminus A^0$. Conversely, if $x \in \overline{A} \setminus A^0$, then for every $r > 0$, $B(x,r) \cap [X \setminus A] \neq \emptyset$ since otherwise $B(x,r) \subset A$ for some r and then $x \in A^0$. Hence $x \in \overline{X \setminus A}$ and $x \in \partial A$.
(c) If $\overline{A} = A$, then in view of (b), $\partial A = \overline{A} \setminus A^0 = A \setminus A^0$. If $\partial A = A \setminus A^0$, then $\partial A \subset A$ and in view of (a), $\overline{A} = A \cup \partial A \subset A$ and so A is closed.
(d) If A is open, then using (b), $\partial A = \overline{A} \setminus A^0 = \overline{A} \setminus A$. Conversely, if $\partial A = \overline{A} \setminus A$, then in view of the second part of (b), $A^0 = A \setminus \partial A = A \setminus [\overline{A} \setminus A] = A \setminus [\overline{A} \cap (X \setminus A)] = (A \setminus \overline{A}) \cup [A \setminus (X \setminus A)] = A$.

10. (a) Let $a \in A$. We have to show that a is an interior point of A. Since B is non-empty, there is $b \in B$. So $(a,b) \in A \times B$. Since $A \times B$ is open, (a,b) is an interior point of $A \times B$ and there exists $r > 0$ such that $B((a,b),r) \subset A \times B$. For any $x \in B(a,r)$, we have

$$d((a,b),(x,b)) = d_X(a,x) + d_Y(b,b) = d_X(a,x) < r,$$

so that $(x,b) \in B((a,b),r) \subset A \times B$. Hence $B(a,r) \subset A$ and a is an interior point of A. Consequently, any point in A is an interior point of A which means that A is open.
(b) Let $x \in X$ be an adherent point of A. We have to show that $x \in A$. There exists a sequence $\{x_n\}$ such that $x_n \in A$ and $x_n \to x$ in X. Since B is non-empty, there is $y \in B$. For the sequence $\{(x_n, y) \in A \times B$, we have

$$d((x_n,y),(x,y)) = d_X(x_n,x) + d_Y(y,y) = d_X(x_n,x) \to 0,$$

showing that (x,y) is an adherent point of $A \times B$. Since $A \times B$ is closed in $X \times Y$, $(x,y) \in A \times B$, that is, $x \in A$, as required.