Some solutions to Problem Set 3.

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2. If A is not bounded i.e., if diam $A = \infty$, then diam $\overline{A} = \infty$ since $A \subset \overline{A}$. Now assume that $a :=$ diam $A < \infty$. If x, y are adherent points of A, then there are sequences $x_n, y_n \in A$ such that $d(x_n, x) \to 0$ and $d(y_n, y) \to 0$. Then

$$d(x, y) \leqslant d(x, x_n) + d(x_n, y_n) + d(y_n, y) \leqslant a + d(x, x_n) + d(y_n, y).$$

Taking the limit as $n \to \infty$ gives $d(x, y) \leqslant a$ for all $x, y \in \overline{A}$, hence diam $\overline{A} \leqslant a$. Since it is also true that $a \leqslant$ diam $\overline{A}$ we conclude that $a =$ diam $\overline{A}$. In general, diam $A \neq$ diam A^0. For example, let $A = (0, 1) \cap \mathbb{Q}$ with the usual metric. Then diam $A = 1$, but $A^0 = \emptyset$.

3. We claim that if $(x, y) \in X \times Y$, then it is an adherent point of $A \times B$. Since $\overline{A} = X$ and $\overline{B} = Y$, the point x is an adherent point of A and y is an adherent point of B. Hence there exist sequences $\{x_n\}$ and $\{y_n\}$ in A and B such that $x_n \to x$ in X and $y_n \to y$ in Y. Consequently, $(x_n, y_n) \to (x, y)$ in $X \times Y$ and (x, y) is an adherent point of $A \times B$ as claimed.

4 (a) The point $f(t)$ has coordinates $(\frac{t}{t^2+1}, \frac{t^2}{t^2+1})$ and so $f(t) = (\frac{t}{t^2+1}, \frac{t^2}{t^2+1})$. Since $g(t) = \frac{t}{t^2+1}$ and $h(t) = \frac{t^2}{t^2+1}$ are continuous from $\mathbb{R}$ to $\mathbb{R}^2$ with the standard metrics, the function $f(t) = (g(t), h(t))$ is continuous. The inverse function $f^{-1} : X \to \mathbb{R}$ is $f^{-1}(x, y) = \frac{x}{1-y}$ for $(x, y) \in X$. Since $g(x, y) = x$ and $h(x, y) = 1 - y$ are continuous from $\mathbb{R}^2 \to \mathbb{R}$ and $h(x, y) \neq 0$ for $(x, y) \in X$, the function $f^{-1}(x, y) = \frac{g(x,y)}{h(x,y)}$ is continuous from X to $\mathbb{R}$.
(b) If $|s_n - s| \to 0$, then $\|f(s_n) - f(s)\| \to 0$ since f is continuous, and so $\rho(s_n, s) \to 0$. If $\rho(s_n, s) \to 0$, then $\|f(s_n) - f(s)\| \to 0$ and since f^{-1} is continuous from X to $\mathbb{R}$ we get, by the definition of continuity, $|f^{-1}(f(s_n)) - f^{-1}(f(s_n))| = |s_n - s| \to 0$. Hence both metrics on $\mathbb{R}$ are equivalent.

5. (a) Since $|F(f) - F(g)| = |f(0) - g(0)| \leqslant d_\infty(f, g)$, the function F is continuous from (X, d_∞) to $\mathbb{R}$.
(b) The function F is not continuous when X is equipped with the metric d_1. Indeed, define $f_n(x) = -nx + 1$ if $0 \leqslant x \leqslant 1/n$ and $f_n(x) = 0$ for $1/n \leqslant x \leqslant 1$. Then $d_1(f_n, 0) = 1/(2n) \to 0$ but $F(f_n) = f_n(0) = 1 \not\to 0 = F(0)$.

6. (a) Assume that f is continuous. Let $A \subseteq X$. Then $\overline{f(A)}$ is a closed subspace of Y, and so $f^{-1}(\overline{f(A)})$ is a closed subspace of X. Since $f^{-1}(\overline{f(A)})$ is closed and $A \subseteq f^{-1}(f(A)) \subseteq f^{-1}(\overline{f(A)})$, we get $\overline{A} \subseteq \overline{f^{-1}(\overline{f(A)})}$. Hence $f(\overline{A}) \subseteq f(f^{-1}(\overline{f(A)})) = \overline{f(A)}$ as required. Conversely, assume that $f(\overline{A}) \subseteq \overline{f(A)}$ for all $A \subseteq X$. Let $B \subseteq Y$ be closed. We have to show that $A := f^{-1}(B)$ is a closed subset of X. Since B is closed and $f(\overline{A}) \subseteq \overline{f(A)} = \overline{f(f^{-1}(B))} \subseteq \overline{B} = B$, we conclude $\overline{A} \subseteq f^{-1}(B) = A$. Hence $A = f^{-1}(B)$ is closed as required.
(b) Assume that f is continuous and let $B \subseteq Y$. Then $f^{-1}(\overline{B})$ is closed since $\overline{B}$ is closed and f is continuous. Since $f^{-1}(B) \subseteq f^{-1}(\overline{B})$, $\overline{f^{-1}(B)} \subseteq \overline{f^{-1}(\overline{B})} = f^{-1}(\overline{B})$ and the result follows. Conversely, assume that $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$ for all $B \subseteq Y$. Let $B \subseteq Y$ be closed. Then $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B}) = f^{-1}(B)$ so that $f^{-1}(B)$ is closed, and f is continuous.

7. By the triangle inequality,

$$d(x, a) \leqslant d(x, y) + d(y, a)$$

for all $x, y \in X$. So

$$d(x, a) - d(y, a) \leqslant d(x, y).$$

But x, y are arbitrary, so

$$d(y,a) - d(x,a) \leqslant d(y,x) = d(x,y).$$

Combining this with the above inequality gives

$$-d(x,y) \leqslant d(x,a) - d(y,a) \leqslant d(x,y), \text{ that is, } |d(x,a) - d(y,a)| \leqslant d(x,y).$$

8. (a) f is uniformly continuous since $|f(x) - f(y)| \leqslant |x - y|$ for all x, y by the mean value theorem.
(b) g is not uniformly continuous since if $x_n = 1 - 1/n$ and $y_n = 1 - 1/(2n)$, then $|x_n - y_n| = 1/2n \to 0$ and $|f(x_n) - f(y_n)| = n \to \infty$.
(c) h is uniformly continuous. Let $\varepsilon > 0$. Choose $\delta < \varepsilon^2$. Recall (Problem Sheet 1, Q4) that $\sqrt{a+b} \leqslant \sqrt{a} + \sqrt{b}$ for all $a, b \geq 0$. Then if $0 \leqslant x \leqslant y$ and $y - x \leqslant \delta$, then $\sqrt{y} = \sqrt{(y-x)+x} \leqslant \sqrt{y-x} + \sqrt{x}$, that is, $0 \leqslant \sqrt{y} - \sqrt{x} \leqslant \sqrt{y-x} \leqslant \sqrt{\delta} < \varepsilon$.
(d) k is not uniformly continuous since if $x_n = \frac{1}{2n\pi}$ and $y_n = \frac{1}{2n\pi + \pi/2}$ then $|k(x_n) - k(y_n)| = 1$ and $|x_n - y_n| = \frac{\pi}{4n\pi(2n\pi + (\pi/2))} \to 0$ as $n \to \infty$.

9. (a) f_n converges pointwise to 0. Further, f_n attains its maximum at $x_n = \frac{2}{n+2}$. Hence, $0 \leq f_n(x) \leq f_n(x_n) = n\left(\frac{2}{n+2}\right)^2\left(\frac{n}{n+2}\right)^n \leq \frac{4}{n}$ for $0 \leq x \leq 1$. Since $\frac{4}{n} \to 0$, the sequence (f_n) converges uniformly to the function $f = 0$.
(b) If $x = 0$, then $f_n(0) = 0$. For $0 < x \leq 1$ $f_n(x) = n^2 x(1-x^2)^n \to 0$ as $n \to \infty$, by a standard limit. So the sequence (f_n) converges pointwise to $f = 0$. The convergence is not uniform since f_n attains its maximum at the point $x_n = \frac{1}{\sqrt{2n+1}}$ and $f_n(x_n) \to \infty$ as $n \to \infty$.
(c) If $x = 0$, then $f_n(0) = 0$. For $x > 0$ we have $f_n(x) = n^2 x^3 e^{-nx^2} = \frac{n^2}{(e^{x^2})^n} x^3 \to 0$ as $n \to \infty$, by a standard limit. Hence, the sequence (f_n) converges pointwise to $f = 0$. The function f_n attains its maximum at $x_n = \sqrt{3/(2n)}$. The sequence does not converge uniformly to $f = 0$ since $f_n(x_n) = n^2 x_n^3 e^{-nx_n^2} = (\sqrt{3/2})^3 e^{-3/2} \cdot \sqrt{n} \to \infty$ as $n \to \infty$.

10. Define $g(x) = \lim f(x_n)$ where (x_n) is a sequence of points in A converging to x. (Since (x_n) is a Cauchy sequence and f is uniformly continuous, it follows that $f(x_n)$ is a Cauchy sequence in $\mathbb{R}$, hence convergent since $\mathbb{R}$ is complete.) We have to show that $g(x)$ is well-defined, that is, it doesn't depend on the choice of a sequence (x_n) converging to x. If (y_n) is another sequence converging to x, then by the triangle inequality $d(x_n, y_n) \to 0$ and this implies that $|f(x_n) - f(y_n)| \to 0$ since f is uniformly continuous. So $\lim f(x_n) = \lim f(y_n)$. Now let $\varepsilon > 0$. Since f is uniformly continuous there is $\delta > 0$ such that $a, b \in A$ and $d(a,b) < \delta$ implies $|f(a) - f(b)| < \varepsilon/2$. Take $x, y \in X$ such that $d(x,y) < \delta/2$. Then there are sequences $(x_n), (y_n) \subset A$ such that $d(x, x_n) \to 0$ and $d(y_n, y) \to 0$. So there is $n \geq N$ such that $d(x, x_n) < \delta/4$ and $d(y_n, y) < \delta/4$. Hence, by the triangle inequality,

$$d(x_n, y_n) \leqslant d(x_n, x) + d(x,y) + d(y, y_n) < \delta$$

for all $n \geq N$ so that $|f(x_n) - f(y_n)| < \varepsilon/2$. From this we conclude that

$$\begin{aligned} |g(x) - g(y)| &\leqslant |g(x) - f(x_n)| + |f(x_n) - f(y_n)| + |f(y_n) - g(y)| \\ &< |g(x) - f(x_n)| + |f(y_n) - g(y)| + \varepsilon/2 \end{aligned}$$

and after taking limits as $n \to \infty$ we get

$$|g(x) - g(y)| \leqslant \varepsilon/2 < \varepsilon$$

if $d_X(x,y) < \delta/2$. Hence g is uniformly continuous. To see that there is exactly one such a function, assume that there are two continuous function $g : X \to \mathbb{R}$ and $h : X \to \mathbb{R}$ such that $f = g = h$ on A. Then $g - h = 0$ on A and since $g - h$ is continuous and A is dense, $g - h = 0$ on X. So $g = h$.